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Yu. A. SHASHKIN

THE EULER CHARACTERISTIC

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Ю. А. Шашкин

ЭЙЛЕРОВА ХАРАКТЕРИСТИКА

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TO THE READER

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Preface

Leonhard Euler (1707-1783) was one of the most prominent mathematicians of the 18th century. He was born in the Swiss town of Basel. When he was twenty he went to live in St. Petersburg, then he moved to Berlin, and then he returned to St. Petersburg. Euler played a notable part in the development of mathematics, mechanics, physics and engineering. He was a pioneer of mathematical investigations in Russia.

In 1758 Euler presented in "Novi Commentarii Academiae Scientarum Petropolitanae" (Proceedings of the Petersburg Academy of Sciences) a proof of the formula

$$V - E + F = 2 \tag{0.1}$$

relating the number of vertices, V, the number of edges, E, and the number of faces, F, of an arbitrary convex polytope.

This booklet is devoted to Euler's formula (0.1) and to its different analogues and applications. Assume, for example, that there is a finite family of straight lines in the plane which intersect in some number V of points ("vertices") and divide the plane into a number F of "faces" and are themselves decomposed into a number E of "edges". Then it turns out that

$$V - E + F = 1. \tag{0.2}$$

Just as (0.1) is true for any convex polytope, (0.2) is true for any family of straight lines in the plane and is independent of both their number and their relative position.

Generally, remarkable is the fact that given some figure, whatever the way we may wish to divide it into parts (faces, edges or vertices) "adjoining" in a certain manner, the alternating sum $V - E + F$ called the Euler characteristic of the figure remains constant in value.

In the first part of this book (Secs. 1 to 7) the Euler characteristic for a straight line, a plane, a three-dimensional space, polygons of different classes, and the boundaries of convex polytopes are calculated. In Secs. 4 and 5 applications of the Euler characteristic to the calculation of the area of a polygon and the sum of its exterior angles are given.

In the second part (Secs. 8 to 12) the Euler characteristic for a figure (for example, a polygon) is defined axiomatically as an "additive function" of that figure. In this respect it resembles the area of a polygon. It is known that to find the area of the union of two polygons it is necessary to subtract from the sum of their areas the area of their intersection. This is the additive property of the area. One of the axioms of the Euler characteristic requires that it should possess a similar property. The other axiom (namely, that of "normalization") distinguishes between the area and the characteristic. The "normalization" of the first of these two functions of a polygon requires that the area of a unit square should equal one. The Euler characteristic is "normalized" so that it equals one on every convex polygon.

Section 9 gives a proof of the existence of the Euler characteristic which satisfies the given axioms and Sec. 10 proves the equivalence of two different definitions of it. The concluding Sec. 12 contains applications of the Euler characteristic to some problems of combinatorial geometry, a new trend in mathematics with which the reader may get acquainted, for example, from [2], [5] and [6] (see References on page 96).

Nothing is said in this book about the topological invariance of the Euler characteristic or about the part it plays in topology. The reader can get information about this from Boltyansky and Efremovich [1].

The author expresses gratitude to I. Ya. Gusak and A. G. Netrobin for help in the work at this book.

Yu. A. Shashkin

1. Euler's Formulas for a Straight Line and a Plane

The simplest version of Euler's formula arises in dividing a straight line L by a finite set of points. If V points are chosen on a straight line, then they divide it into $V - 1$ line segments and two rays, i.e. into $V + 1$ parts all together. Denoting the number of parts by E we have

$$V - E = -1. \tag{1.1}$$

It is *Euler's formula for a straight line.* It shows that the difference $V - E$ is constant, i.e. independent of both the number of points chosen and their position and is thus a property of a straight line itself.

Let us now proceed to a plane Q and try to obtain for it Euler's formula similar to (1.1). For a plane the problem is more complicated and more interesting than it is for the straight line, for division is now carried out by a finite family of straight lines, and these may be arranged in different ways in the plane. For example, two straight lines may either intersect or be parallel. For three straight lines, there are four cases of relative position. Three straight lines may be all parallel. Two straight lines may be parallel, and the third may intersect them. Each pair of straight lines may have a point in common but there may be no point common for all the three. All the three straight lines may pass through a single point. The various cases of arrangement of four straight lines are depicted in Fig. 1. Their verbal description would be encumbered with difficulties. It would be possible to consider arrangements of five, six or more straight lines. Increasing the number of straight lines quickly increases the number of ways for arranging them.

Every family of straight lines decomposes the plane into parts called *partition faces*; their number will be denoted by F. By *partition vertices* we mean intersection points of the given straight lines, and by *partition edges* we mean the parts into which the straight lines are divided by the vertices. The number of vertices and the number of edges

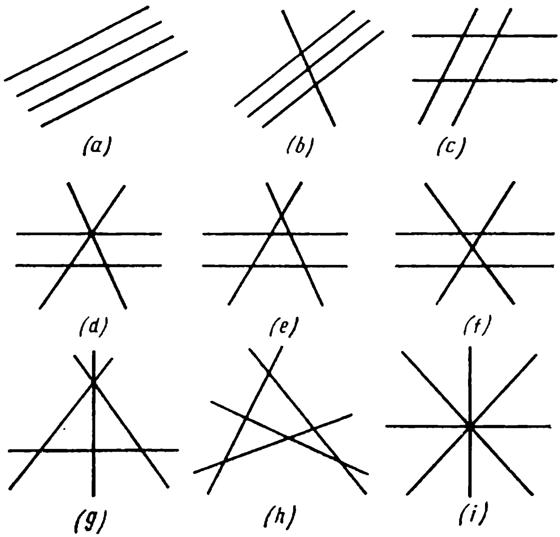

Fig. 1

will be denoted by V and E respectively. A partition may have no vertices (and then $V = 0$). This will be the case if and only if every two straight lines are parallel. The lines themselves would be the edges of such a partition.

It turns out that the numbers V, E and F are connected by the relation

$$V - E + F = 1. \tag{1.2}$$

This is *Euler's formula for a plane.* It shows that the alternating sum $V - E + F$ is constant, i.e. it is independent of both the number of straight lines and their relative position. Consequently, Euler's formula expresses a property of the plane Q itself.

Let us prove (1.2) for a partition of n straight lines. We first make some remarks.

First, in the case where a partition has no vertices (1.2) is obvious, since then $V = 0$, $E = n$, and $F = n + 1$.

Second, we prove the following lemma which will turn out to be helpful in other cases as well.

Lemma 1. *Given a finite number of straight lines in the plane, it is possible to draw a new straight line which is not parallel to any one of those given.*

Proof. Indeed, suppose straight lines $L_1, \ldots, L_m$ are given and O is some point on L_1. We draw through O a straight line $L'_i \parallel L_i$ $(i = 2, \ldots, m)$ and let φ_i denote the angle between L_1 and L'_i, assuming $0 \leqslant \varphi_i \leqslant 90°$. If $\varphi_i = 0$ for all i, i.e. if all the given lines are parallel, then the required line will be any one that is not parallel to L_1. Otherwise we choose the smallest positive angle. Let it be φ_2, for example. A straight line L which passes through O and makes with L_1 a positive angle smaller than φ_2 is the required one. Lemma 1 is proved.

Third, in order to prove (1.2) we must find the expressions for two of the three quantities, V, E and F, assuming the third quantity to be known. Suppose that the number of vertices V (and, of course, the number of straight lines, n) is known. However, the numbers F and E are not determined uniquely by the numbers n and V. For example, partitions (c) and (g) (Fig. 1) have the same number of vertices, $V = 4$, but different numbers of faces and edges. A look at these partitions shows the difference in the "structure" of the vertices; in (c) two straight lines pass through each vertex and in (g) there is a vertex through which three straight lines pass. To take this difference into account we introduce the following definition. The *multiplicity* of a vertex of a partition is the number of straight lines of a given family that pass through it. Thus the multiplicity of any vertex is a natural number not less than two. It is natural to expect that the numbers F and E are defined if besides n and V the multiplicities of all the vertices are considered to be given. We shall see that this is indeed the case.

The calculation of edges and faces of the partition will be carried out by the method of a "moving" line.

Suppose $L_1, \ldots, L_n$ are given straight lines and $A_1, \ldots, A_V$ are the vertices (Fig. 2, where $n = 5$ and $V = 7$). Let us draw an auxiliary line through each pair of vertices and denote these lines by $M_1, \ldots, M_k$. All the given straight lines, $L_1, \ldots, L_n$, are among them, of course. (The "superfluous" auxiliary straight lines, i.e. those different from $L_1, \ldots, L_n$, are not represented in Fig. 2 not to overload it; the reader will verify that they should have been drawn through the pairs of points A_1A_5, A_2A_5, A_2A_7, A_3A_4 and A_6A_7.) Using Lemma 1, let us now draw an auxiliary straight line L_0 which is not parallel to any of the straight lines $M_1, \ldots, M_k$.

We shall assume that, first, L_0 is horizontal, and, second, is below all the vertices $A_1, \ldots, A_V$. It follows that for each pair of vertices A_i and A_j, their distances from L_0 are different*. In what follows this fact will be expressed by saying that "the vertices are all at different heights", meaning their height above the level of L_0. We

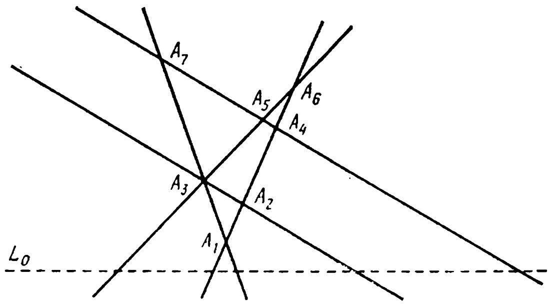

Fig. 2

shall assume that the vertices are numbered in increasing order of their heights, i.e. let A_1 be the lowest vertex and A_2 lie above A_1 but below A_3 and so on, and, finally, A_V be the uppermost vertex.

The "moving" line L will be horizontal, and initially coincide with L_0 and then it will move along the plane. The line L may be used to calculate the number of the edges of the partition. Since it intersects all the lines $L_1, \ldots, L_n$, each at "its own" single point, in its initial position it meets n edges. Now we make L move up the plane parallel to itself. Until it meets the lowest vertex A_1, the number of edges it crosses remains unaltered and equal to n. After passing through A_1 the number changes, new edges appear the number of which equals the multiplicity α_1 of the vertex A_1. The total number of edges the line L meets thus far becomes $n + \alpha_1$ and remains so until it meets the next vertex, A_2. If A_2 has a multiplicity α_2, then after L passes through A_2 the number of edges already met by L again rises and becomes $n + \alpha_1 + \alpha_2$ and so on. Finally, after L passes through the last, the uppermost vertex, A_V, of multiplic-

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* Indeed, if A_i and A_j are the same distance from L_0, then a line passing through these vertices will be parallel to L_0. This is impossible, however, by the construction of L_0.

ity α_V, this number becomes $n + \alpha_1 + \alpha_2 + \ldots + \alpha_V$. So the total number of partition edges is

$$E = n + \alpha_1 + \alpha_2 + \ldots + \alpha_V$$

or in shorter form

$$E = n + \sum_{i=1}^{V} \alpha_i. \tag{1.3}$$

The number of faces of a partition is found as follows. In its initial position L is divided by $L_1, \ldots, L_n$ into $n + 1$ parts, each lying in its own face and thus counting that face. Hence in its initial position the line L meets $n + 1$ faces and that number remains constant until L reaches the vertex A_1. As we have seen, new edges appear in the number α_1 after L passes through A_1. It is clear that the number of new faces met by L by that moment is $\alpha_1 - 1$.* The total number of faces already met is therefore $1 + n + \alpha_1 - 1$. After the line L passes through A_2 the total number increases by $\alpha_2 - 1$ and so on. Finally, when L crosses the last uppermost vertex, A_V, the total number of faces will increase by another $\alpha_V - 1$. Therefore

$$F = n + 1 + (\alpha_1 - 1) + (\alpha_2 - 1) + \ldots + (\alpha_V - 1)$$

or

$$F = 1 + n + \sum_{i=1}^{V} (\alpha_i - 1) = 1 + n + \sum_{i=1}^{V} \alpha_i - \sum_{i=1}^{V} 1$$

$$= 1 + n - V + \sum_{i=1}^{V} \alpha_i. \tag{1.4}$$

Here $\sum_{i=1}^{V} 1$ denotes the sum of ones taken over all the vertices and is therefore equal to V.

So we have expressed the number of edges and that of faces in terms of the number of vertices and their multiplicities. It can be seen from (1.3) and (1.4) that the numbers E and F depend only on the sum of multiplicities and are independent, for example, of the order in which they appear. Now (1.3) and (1.4) immediately yield Eu-

---

* These new faces are between the α_1 new edges.

ler's formula

$$V-E+F=V-n-\sum_{i=1}^{V}\alpha_i+1+n-V+\sum_{i=1}^{V}\alpha_i=1$$

Notice that the proof ofthis formula can be made simpler if we do not try to derive an explicit expression for the numbers E and F. Namely, let all the vertices of the partition lie within a horizontal strip of width H (Fig. 3).

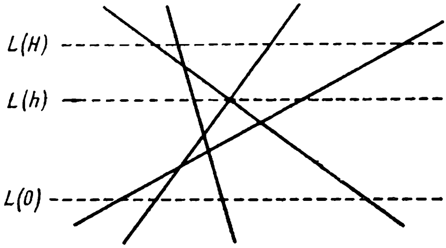


Fig. 3

Let L (0) be the lower boundary of the strip, i.e. the line L in its initial position, L (H) be the upper boundary of the strip, i.e. the final position of L, and L (h) be the line L at a distance h from L (0). Let E (h) be the number of edges the moving line L met by the time it has occupied the position L (h); V (h) and F (h) have a similar meaning. The values of E (h), V (h) and F (h) change depending on h; we must prove that their alternating sum $S(h)=V(h)-E(h)+F(h)$ assumes the value of 1 when $h=H$. If $h=0$, then, as we have seen, $V(h)=V(0)=0$, $E(0)=n$, $F(0)=1+n$, and therefore $S(0)=1$. When L passes through A_i the number E (h) increases by α_i, F (h) increases by α_i-1, and V (h) by 1, therefore S (h) remains unaltered in value. In particular, $S(H)=1$, which was to be proved.

Problems

1. Prove that for any partition of a straight line by a finite number V of points the sum $V+E$ is odd.

2. Prove that for any partition of the plane by a finite number of lines the sum $V+E+F$ is odd.

3. Straight lines in the plane are said to be *in general position* if no two of them are parallel and no three of them have a point in common. Prove that for any natural number n in the plane there are n straight lines in general position.

4. Show that if a plane is partitioned by n straight lines in general position, then

$$E=n^2, \quad F=1+n+\frac{n(n-1)}{2}.$$

5. Prove Euler's formula (1.2) by the method of a moving line, assuming that the line is parallel to one or several lines of the family.

6. For a family of lines in general position, prove Euler's formula by induction on the number of straight lines.

7. A plane figure is said to be *bounded* if it lies in a circle of some (possibly, very large) radius, and *unbounded* otherwise. Find the number E_1 of bounded edges (i.e. line segments) and the number E_2 of unbounded edges (i.e. rays), as well as the number F_1 of bounded faces and the number F_2 of unbounded faces for partition of a plane having vertices. Show that

$$V-E_1+F_1=1.$$

8. Clearly a partition of a plane usually has unbounded faces and unbounded edges. When does it have bounded faces (bounded edges)?

2. What Is the Euler Characteristic?

Euler's formulas for a straight line and a plane which have been proved in Sec. 1 are a manifestation of the following remarkable general fact. Let Φ be a figure divided some way into parts called *vertices*, *edges* and *faces*. Denote the number of vertices, the number of edges and the number of faces of the partition by V, E, and F, respectively. For all parts of the partition we shall also use the term *cell*, i.e. instead of the words "face, edge or vertex of a partition" we shall use "cells of a partition". It turns out that regardless of the way of decomposing Φ into cells the alternating sum $V-E+F$ remains constant in value or invariant with respect to the way of partitioning. This sum is called the Euler characteristic of the figure and denoted by the Greek letter χ (read kai). Thus by definition

$$\chi(\Phi)=V-E+F.$$

As shown above, the Euler characteristic of a straight line is —1 and that of a plane is 1.

The order of terms in the alternating sum $V-E+F$ defining the Euler characteristic is not accidental, it depends on the dimension of the cells corresponding to these terms. In other words, V denotes the number of

zero-dimensional cells of the partition, i.e. cells having dimension zero; E denotes the number of one-dimensional cells and F the number of two-dimensional cells. Notice that zero-dimensional cells (or vertices) are points, one-dimensional cells (or edges) are most often rectilinear segments and two-dimensional cells (faces) are convex polygons.

The definition of the Euler characteristic we have given here needs to be refined, we should name the classes of the figures we are considering, explain what is meant by the cell in each particular case and how the partition of a figure into cells is defined, i.e. how different cells "adjoin". It is to these questions that almost the whole of Sec. 2 is devoted.

So let us consider some classes of figures for which the Euler characteristic can be defined.

Let $A_1, \ldots, A_n$ be different points in the plane. A *nonclosed broken line* with vertices at those points is defined as a union of rectilinear segments $A_1A_2, A_2A_3, \ldots, A_{n-1}A_n$ called *edges*. If $n \geqslant 3$ and if the first and the last vertex coincide (i.e. $A_1 = A_n$) and all the other vertices are different, then the broken line is said to be *closed*. Two edges of a broken line having a vertex in common are considered to be *adjacent*. A broken line (whether closed or not) is said to be *simple* if no two of its edges that are not adjacent have points in common. We call a simple closed broken line a *circuit*.

It is clear that the Euler characteristic $\chi(L) = V - E$ is 1 for a nonclosed broken line L and 0 for a closed one. It can also be easily verified that $\chi(L)$ remains unaltered if we introduce an arbitrary number m of new vertices inside some edge, thus decomposing it into a new number of edges $m + 1$, or conversely, if we replace by a single edge several successive edges lying on the same line.

A *graph* is a figure G consisting of a finite number of vertices (lying in a plane or in space) and rectilinear segments connecting some pairs of vertices. These segments are called *edges* of the graph. It is clear that in particular any broken line is a graph. The Euler characteristic of a graph is the difference $V - E$; it is invariant in the same sense as for a broken line. It may happen that the graph G has no edges at all and only consists of n vertices; in this case $\chi(G) = n$. Examples of graphs are shown in Fig. 4, their vertices represented by small light

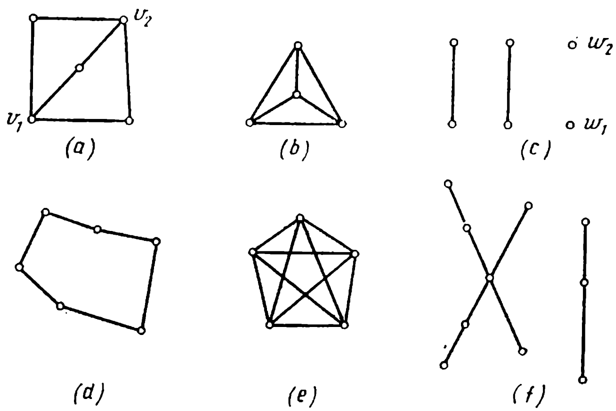


Fig. 4

circles. It can be seen from the figure that some edges may have "extra" points of intersection that are not vertices. The reader will find that the Euler characteristics of graphs (a), (b), (c), (d), (e) and (f) in that figure are —1, —2, 4, 0, —5 and 2, respectively.

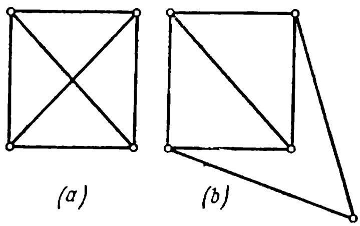


Fig. 5

The *degree* of a vertex of a graph is the number of edges which join it to the other vertices. In Fig. 4, for example, the vertices v_1 and v_2 of graph (a) are of degree 3; the other vertices of the graph are of degree 2. Graph (c) has two vertices of degree 0 (namely, w_1 and w_2) and four vertices of degree 1. Each vertex of graph (b) is of degree 3 and each vertex of graph (e) is of degree 4. Vertices of degree 0 are also termed *isolated vertices*.

A graph is said to be *connected* if any two of its vertices can be joined by a nonclosed broken line consisting of edges.

A graph is said to be *embedded in the plane* if it can be drawn in the plane in such a way that its edges intersect only at vertices. Such, for example, is graph (a) represented in Fig. 5 (the so-called *complete* graph with four vertices) for it could be "redrawn" so that the extra inter-

sections of edges disappear (Fig. 5*b*). In doing this, however, we had to replace one edge of the graph by two edges and to introduce a new vertex, and in this case the graph is usually considered to be "unchanged"; at any rate the value of the Euler characteristic of the graph remains unchanged. Note also that Fig. 4*b* represents the "same" graph as Fig. 5*a* does, with no extra inter-

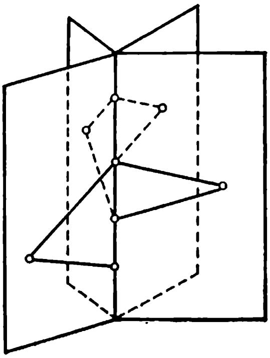


Fig. 6

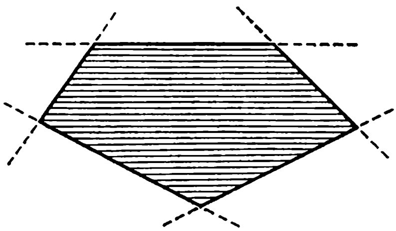


Fig. 7

sections of edges and no new vertices. In general, all the graphs of Fig. 4 except (e) can be embedded in the plane. On the other hand, graph (e) (called the *complete* graph with five vertices) cannot be embedded in the plane even if we introduce new vertices (see Problem 13).

It is of interest to note that every graph can be sketched in space without extra intersections of edges. We prove it. Let a graph G have V vertices and E edges. We take in space a "little book with E sheets" (Fig. 6 where $E = 4$), and mark on its "back" V points (graph's vertices). We associate each of the E edges of the graph with a distinct sheet of the book and sketch on it the edge as a broken line consisting of two line segments with no extra intersections of edges; therefore we had to replace each edge by two new edges and one new vertex. We should notice that every graph can be sketched in space without introducting new vertices and without breaks in the edges.

We now proceed to the next class of figures, that of plane polygons, considering first the simplest of them, convex polygons.

Every straight line divides the plane into two half-planes. It is assumed that the straight line itself is contained in each of them. In other words, we assume that both half-planes are closed. A *convex polygon* is an intersection of a finite number of half-planes provided that it is, first, *bounded*, i.e. it is contained in a circle of finite radius, and, second, *two-dimensional*, i.e. contains a circle of nonzero radius (Fig. 7). The latter requirement is equivalent to the fact that a convex polygon is in no straight line.

Let us now define a general (not necessarily convex) polygon.

A *polygon* is a plane figure M consisting of a union of a finite number of convex polygons so that the following two conditions hold:

(1) any two convex polygons either have no points in common at all or have only a vertex in common or a side in common;

(2) a figure M is *connected*, i.e. any two of its points can be joined by a simple nonclosed broken line which is entirely in M.

The latter condition means that the polygon does not break down into separate disconnected parts.

It is clear from the definition what is to be meant by a partition of a polygon M into cells. The convex polygons of which the polygon M consists are the faces of the partition, the sides of the convex polygons are the edges of the partition, and the vertices of the polygons are the vertices of the partition. Figure 8 gives examples of polygons and their partitions. Clearly, every polygon allows different representations as a union of convex polygons and has therefore different partitions.

Now the concept of Euler characteristic acquires an exact meaning. It will be shown in the next section that it is independent of the choice of the way of partitioning.

Interior and boundary points are distinguished in a polygon. A point of a polygon is said to be *interior* if it is possible to indicate a circle of some (even if very small) radius with centre at that point, which is entirely in the polygon. A point of a polygon is said to be *boundary* if a circle of any radius with centre at that point contains both points of the polygon itself and points of a complement with respect to the plane (i.e. points of the

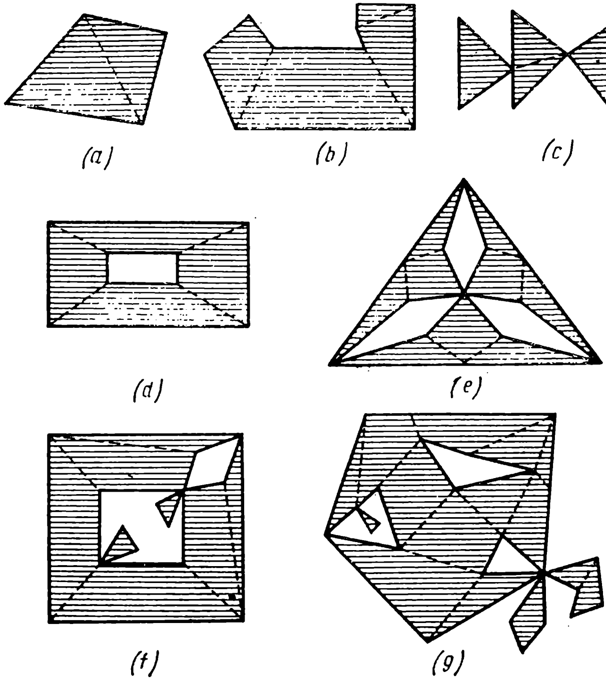


Fig. 8

plane that do not belong to the polygon). Points A and B shown in Fig. 9 are the interior points of the polygon, C and D are the boundary points, and E and F are the points of the complement. The set of all boundary points of a polygon is called the *boundary*.

We prove that the *boundary of a polygon M is the union of a finite number of circuits*. That boundary is the graph G. We first show that each vertex of the graph is of even degree. Indeed, let A be a vertex of G. Draw a circle with centre at the point A of radius so small that it intersects only the graph edges emanating from A. Let $A_1, A_2, \ldots, A_n$ be successive points of the intersection of the circle with the edges (Fig. 10). We now move along the circle from A_1 to A_2, then to A_3 and so on. Moving from A_1 we move from the polygon M into its complement, and, conversely, in passing through A_2 we enter the polygon. Since passing through the last point, A_n,

we again enter M, and the ins and outs alternate, the number n must be even.

We shall now move along the graph edges, beginning the movement from some vertex and traversing no edge twice. Since all the vertices are of even degree, entering any vertex we have a possibility of leaving it. On the other hand, since the graph G has a finite number of edges and vertices, we must get to such a vertex where we have already been before. In this way we obtain a circuit. We remove the circuit from the graph G. The new graph will

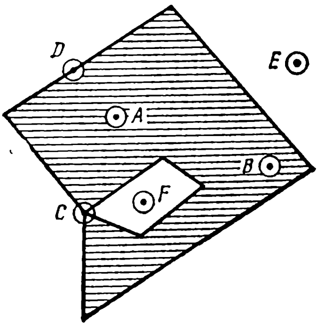


Fig. 9

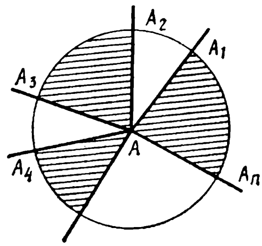


Fig. 10

again have only even degrees of vertices. Tracing the indicated round-trip path several times we obtain the entire graph G, i.e. the boundary of the polygon M, as a union of a finite number of circuits, which was to be proved.

A polygon is said to be *simple* if its boundary consists of a single circuit. Such, for example, are all convex polygons as well as a polygon (b) in Fig. 8. It could be proved (but we shall not do it) that the complement of a simple polygon (with respect to the plane) is connected. Not only simple polygons possess this property (see, for example, Fig. 8*c*).

If the complement of a polygon is disconnected, then it consists of several connected pieces called *components* (Fig. 8*d*, *e*, *f*, *g*). One component is always unbounded, the rest are bounded. These bounded components are termed *holes*.

We prove that *every hole F together with its boundary is a polygon*. To do this it suffices to see that the hole can be represented as a union of convex polygons. Let us draw straight lines through all the segments of the boundary

of F (Fig. 11). Then we obtain some partition of the entire plane. All the bounded faces of that partition are convex polygons. It should be noted that all interior points of each of these polygons lie entirely either in the hole F or in its complement, and, consequently, the hole F (together with its boundary) is equal to the union of the polygons of the former of these two classes, which was to be proved.

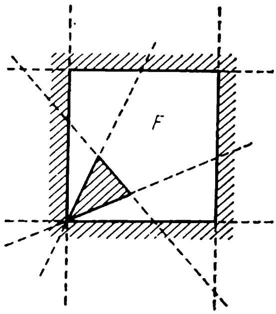


Fig. 11

A hole of a polygon is said to be *simple* if its boundary is a circuit. In the following three Sections 3, 4 and 5, we shall consider simple polygons and also polygons obtained from simple ones by "cutting" a finite number of simple holes. This, in particular, excludes polygons (c), (f), and (g) of Fig. 8 from our consideration.

A remark about a partition of the polygon M into cells should be made: it is sometimes more convenient to consider the *open* convex polygons of which M is made up (i.e. polygons without their boundaries) to be the faces of the partition and the *open* sides of those convex polygons (sides without their ends) to be the edges of the partition. Viewed in this way, different cells of the partition have no points in common. We shall use partition in this sense only once, in Sec. 7 (p. 49).

Problem

9. Given an arbitrary partition of the plane by a finite number of straight lines. Let M be the union of all its bounded faces. Prove that the figure M is connected and hence a polygon. Prove that the polygon M is either simple or equal to the union of two simple polygons having a single point in common.

3. The Euler Characteristic for Polygons

We proceed to calculate the Euler characteristic for polygons applying the method of a moving straight line. In this connection it will be assumed throughout this section that vertices of the partition of the polygon are all at different heights. As before, this can be accomplished by using Lemma 1.

We first consider the case of a simple polygon. We introduce the following classification of its vertices. A vertex v is said to be *protruding upwards* if the interior angle of the polygon at the vertex is less than π (in what follows all the angles are measured in radians) and if both vertices adjacent to it are below v. A vertex w is said to be *entering downwards* if the interior angle of the polygon at that vertex is greater than π and if both adjacent vertices are above w. Vertices of these two classes are called *singular*, all the other vertices of the polygon being termed *ordinary*. The reason for the choice of these names will become clear from what follows. Protruding upwards in Fig. 12 are the vertices v_4 and v_9, the vertex v_2 being entering downwards, the remaining six vertices are ordinary. Let α denote the number of vertices protruding upwards and β the number of vertices entering downwards.

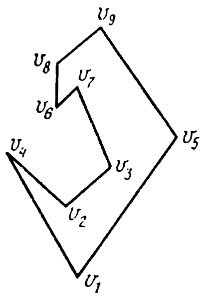


Fig. 12

Lemma 2. *For every simple polygon whose vertices are all at different heights we have*

$$\alpha - \beta = 1 \tag{3.1}$$

Proof. We use induction on the number of sides of a polygon. For a triangle, (3.1) is obvious since every triangle has one vertex protruding upwards* and no other singular vertices. Notice that this is also true for all convex polygons. We prove (3.1) for a simple polygon M with n sides, assuming that it has already been proved for all simple polygons with the number of sides less than n. We first notice that every simple polygon M can be cut by a diagonal lying inside M into two simple polygons, each with a smaller number of sides than M has. Indeed, let A be the lowest vertex of M and B and C the vertices adjacent to it (Fig. 13). Join the points B and C by a diagonal. If there are no other vertices of the polygon M either on the segment BC or inside the triangle ABC, then the diagonal BC gives the required partition. If, however, there are such vertices, then we take the

* Recall that the vertices are all at different heights.

lowest of them; let it be the point D. In this case the required partition of M is given by its diago al AD.

We now return to the proof of (3.1). Let the diagonal AB cut M into two simple polygons M_1 and M_2 and hence be their common side (Fig. 14). To simplify the proof we first assume that the segment AB is vertical, A being its lower end and B its upper end; it will be shown

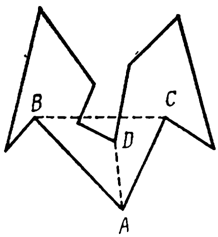


Fig. 13

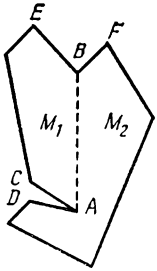


Fig. 14

later what is to be changed in the proof if this condition fails to hold. Also suppose that M_1 adjoins AB on the left and M_2 adjoins it on the right.

If the points A and B are the adjacent vertices of each of the polygons, M_1 and M_2, then denote by C the vertex of M_1 which is a second one adjacent to A and by E the vertex of M_1 which is a second one adjacent to B. Similar in meaning are D and F for the vertices of M_2 (Fig. 14). "Exceptions" are possible, however, where the points A and B (or one of them) are not vertices of one of the polygons M_1 and M_2 (polygon (4) in Fig. 15 and polygon (5) in Fig. 16). Then the notation is indicated in the figures. It may also happen that $C = E$ or $D = F$.

We assign to each vertex v of the polygon M a number $f(v)$ according to the following rule:

$$f(v) = \begin{cases} 1 \text{ if } v \text{ protrudes upwards,} \\ -1 \text{ if } v \text{ enters downwards,} \\ 0 \text{ otherwise.} \end{cases}$$

This means that we define the function f on the vertices of the polygon M. For example, in Fig. 14 this function assumes the value of 1 for the vertices E and F, the value of -1 for the vertices A and B, and the value of 0 for

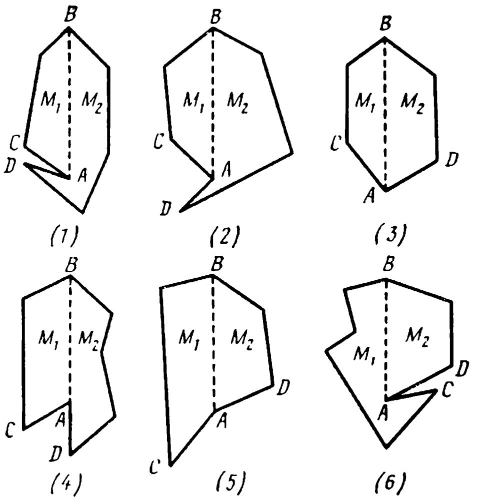


Fig. 15

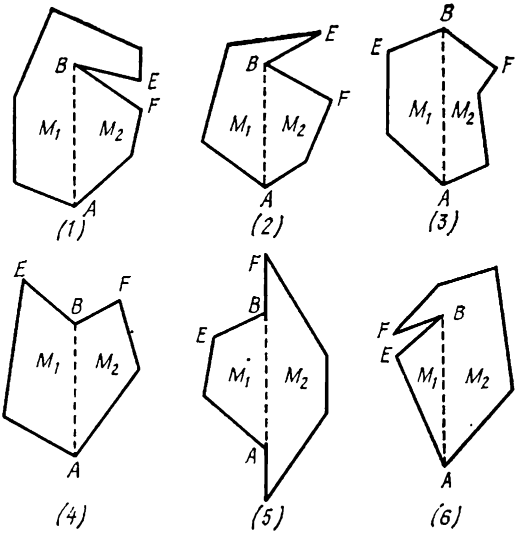


Fig. 16

all the other vertices. We define the functions f_1 and f_2 for the vertices of M_1 and M_2 using the same rule as that for f. The indices 1 and 2 in f_1 and f_2 precisely show that these functions correspond to the polygons M_1 and

M_2. In the case where, for example, the point A is not a vertex of M_1, we nevertheless assume by definition $f_1(A) = 0$. Thus f_1 and f_2 are necessarily defined at A and B too. It can be easily seen that if v is a vertex of M_1 other than A and B, then $f_1(v) = f(v)$; similarly $f_2(v) = f(v)$ for all vertices v of M_2 other than A and B.

Equation (3.1), which is to be proved, can be written as:

$$\sum f(v) = 1, \tag{3.2}$$

where the summation is taken over all the vertices v of M. Since each of the polygons M_1 and M_2 has the number of vertices less than n, under the induction hypothesis we have

$$\sum f_1(v) = 1 \tag{3.3}$$

for the first of them and

$$\sum f_2(v) = 1 \tag{3.4}$$

for the second, where the summation is taken over all the vertices of M_1 and M_2 respectively. We add equations (3.3) and (3.4) term by term and add the same terms on both sides, and as a result we get

$$\begin{aligned}\sum f_1(v) - f_1(A) - f_1(B) + \sum f_2(v) - f_2(A) - f_2(B)\\ + f(A) + f(B) = 2 - f_1(A) - f_1(B) - f_2(A) - f_2(B)\\ + f(A) + f(B).\end{aligned} \tag{3.5}$$

The left-hand side of (3.5) equals $\sum f(v)$, i.e. the sum of the values of the function f over all the vertices of the polygon M. To prove (3.2) it is therefore sufficient to establish that

$$f_1(A) + f_1(B) + f_2(A) + f_2(B) - f(A) - f(B) = 1. \tag{3.6}$$

We prove that

$$f_1(A) + f_2(A) - f(A) = 0, \tag{3.7}$$

$$f_1(B) + f_2(B) - f(B) = 1 \tag{3.8}$$

from which we at once have (3.6) and at the same time the validity of Lemma 2.

Let φ_1 be the value of the angle BAC, and φ_2 the value of the angle BAD (Figs. 14 and 15). We assume that

these angles are measured on the segment AB in the positive direction, i.e. counterclockwise. Then $\varphi_1 < \varphi_2$. Since AB (except for its ends) is inside the polygon M, the values of 0 and 2π for φ_1 and φ_2 are "forbidden". Further, since the vertices of the polygon are all at different heights, the values of $\pi/2$ and $3\pi/2$ are also "forbidden" for these angles. Thus, to prove (3.7) it is necessary to consider the six cases presented in Table 1 and depicted in Fig. 15.

TABLE 1

Cases	Inequalities for angles	$f(A)$	$f_1(A)$	$f_2(A)$
1	$0 < \varphi_1 < \varphi_2 < \pi/2$	-1	0	-1
2	$0 < \varphi_1 < \pi/2 < \varphi_2 < 3\pi/2$	0	0	0
3	$0 < \varphi_1 < \pi/2 < 3\pi/2 < \varphi_2 < 2\pi$	0	0	0
4	$\pi/2 < \varphi_1 < \varphi_2 < 3\pi/2$	0	0	0
5	$\pi/2 < \varphi_1 < 3\pi/2 < \varphi_2 < 2\pi$	0	0	0
6	$3\pi/2 < \varphi_1 < \varphi_2 < 2\pi$	-1	-1	0

Let us first consider carefully the case where $0 < \varphi_1 < \varphi_2 < \pi/2$. In this case points C and D are above the point A and the interior angle CAD of the polygon M is greater than π (it is $2\pi - (\varphi_2 - \varphi_1)$). Therefore the vertex A of M enters downwards and hence $f(A) = -1$. Similarly, points B and D are above A and the interior angle BAD of the polygon M_2 is greater than π (it is $2\pi - \varphi_1$). Therefore the vertex A of M_2 enters downwards and hence $f_2(A) = -1$. On the other hand, vertices B and C of M_1 are above its vertex A and in addition its interior angle BAC is less than π (it is φ_1). Therefore the vertex A is ordinary for M_1 and hence $f_1(A) = 0$. Consequently, (3.7) is satisfied for the first case.

The values of $f(A)$, $f_1(A)$ and $f_2(A)$ in the remaining five cases are listed in Table 1. It can be seen from the table that (3.7) always holds.

Let ψ_1 be the value of the angle ABF and ψ_2 the value of the angle ABE (Figs. 14 or 16). These angles with the vertex at the point B are measured on the segment BA in the positive direction. Therefore $\psi_1 < \psi_2$. As before, the values of 0, $\pi/2$, $3\pi/2$ and 2π are forbidden for these angles. Consequently, to prove (3.8) it is necessary to

consider the six cases represented in Fig. 16. The inequalities relating the angles ψ_1 and ψ_2 in each of these cases and the values of f, f_1 and f_2 at the point B are given in Table 2. It can be seen from the table that (3.8) always holds.

TABLE 2

Cases	Inequalities for angles	$f(B)$	$f_1(B)$	$f_2(B)$
1	$0<\psi_1<\psi_2<\pi/2$	0	0	1
2	$0<\psi_1<\pi/2<\psi_2<3\pi/2$	0	0	1
3	$0<\psi_1<\pi/2<3\pi/2<\psi_2<2\pi$	1	1	1
4	$\pi/2<\psi_1<\psi_2<3\pi/2$	-1	0	0
5	$\pi/2<\psi_1<3\pi/2<\psi_2<2\pi$	0	1	0
6	$3\pi/2<\psi_1<\psi_2<2\pi$	0	1	0

Thus Lemma 2 is proved for the vertical position of the segment AB. Now let AB make an angle ω, $0 \leqslant \omega < \frac{\pi}{2}$ with the vertical straight line. In this case each of the angles φ_1, φ_2, ψ_1 and ψ_2 is "forbidden" to assume the values of 0, 2π, $\frac{\pi}{2}+\omega$, $\frac{3\pi}{2}+\omega$. Otherwise the proof remains the same.

Theorem 1. *The Eulei characteristic of a simple polygon is* 1.

Proof. A simple polygon is assumed to be given with some partition. We place it so that the vertices of the partition are all at different heights. For the validity of the theorem this assumption is not essential, however. We number the vertices of the partition, $v_1, \ldots, v_V$, in an increasing order of their heights, i.e. so that the vertex v_1 is the lowest, v_2 is above v_1, and so on. It follows that if an edge of the partition joins the vertices v_i and v_j, then one of the vertices is the upper end of the edge and the other is the lower end. Similarly, each face of the partition has a unique lowest vertex. Let E_i ($i = 1, \ldots, V$) denote the number of edges for which the point v_i serves as the lower end and F_i the number of faces for which v_i serves as the lowest vertex. Hence the total number of the edges of the partition is

$$E = E_1 + E_2 + \ldots + E_V \tag{3.9}$$

and the total number of faces is

$$F = F_1 + F_2 + \dots + F_V. \qquad (3.10)$$

We find for each vertex v_i a relation between the numbers E_i and F_i. Let M_i be a polygon formed by all the faces (and edges) having a point v_i as the lowest vertex (Figs. 17 and 18 where the polygons M_i are shaded). A polygon M_i may be "singular", i.e. contain not only faces and their sides but also such edges that are not contained in the boundary of some face (in Figs. 17 and 18 such edges are represented by heavy lines). Consider three cases.

1. The point v_i is an ordinary vertex of a polygon M or its interior point (then we say for simplicity that v_i is an ordinary vertex of the partition). Since every face is convex, in this case the horizontal straight line L passing just above the vertex v_i (the "moving" line) intersects the polygon M_i in one segment (Fig. 17). Therefore

$$E_i - F_i = 1. \qquad (3.11)$$

2. The vertex v_i of the polygon M protrudes upwards (Fig. 18*a*). In this case obviously

$$E_i = F_i = 0. \qquad (3.12)$$

3. The vertex v_i of the polygon M enters downwards (Fig. 18*b*, *c*). In this case the straight line L intersects the polygon M_i in two separate segments (which may degenerate into points). Therefore

$$E_i - F_i = 2. \qquad (3.13)$$

It is equations (3.11) to (3.13) that allow us to distinguish between ordinary and singular vertices of a partition.

To find the Euler characteristic we break down the alternating sum $V - E + F$ over the vertices using equations (3.9) and (3.10):

$$\chi(M) = V - E + F = (1 - E_1 + F_1) + (1 - E_2 + F_2) + \dots + (1 - E_V + F_V).$$

In view of equations (3.11) to (3.13) the expression $1 - E_i + F_i$ is zero for every ordinary vertex, unity for every vertex protruding upwards and minus unity for every vertex entering downwards. Therefore $\chi(M) = \alpha - \beta$ or by Lemma 2 $\chi(M) = 1$, proving Theorem 1.

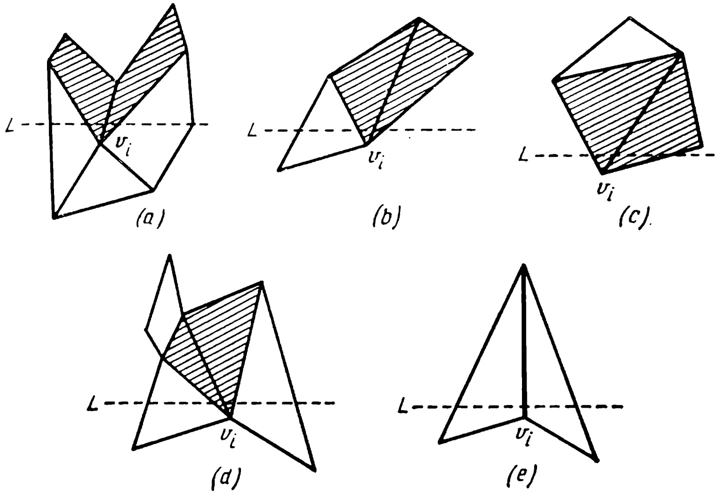


Fig. 17

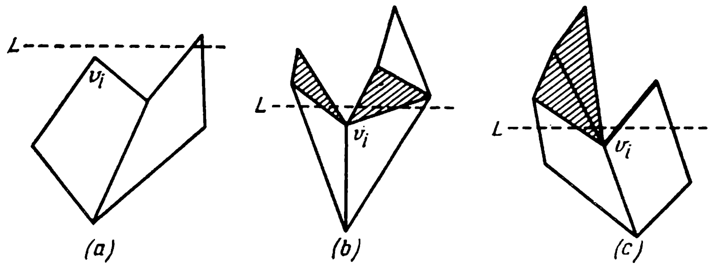


Fig. 18

Corollary. *The Euler characteristic for a simple open polygon (i.e. a polygon whose boundary vertices and edges are removed) is* 1.

Proof. The proof of the corollary follows from Theorem 1 and from the fact that the Euler characteristic for a boundary of a simple polygon is zero.

Theorem 2. *The Euler characteristic for a polygon M having n holes is* $1 - n$.

Below we shall give the proof for a special case of the theorem. The general case is to be proved in Sec. 12. We first introduce definitions and prove a lemma.

Let M be again a simple polygon. We say that the vertex v protrudes downwards if the interior angle of the

polygon at the vertex v is less than π and if the adjacent vertices are both above v. In Fig. 12 v_1 and v_6 are such vertices. We say that the vertex w enters upwards if the interior angle of the polygon at the vertex is greater than π and if the adjacent vertices are both below w. In Fig. 12 v_7 is such a vertex. Denote by γ the number of vertices of a simple polygon that protrude downwards and by δ the number of the vertices that enter upwards.

Lemma 3. *For every simple polygon whose vertices are all at different heights, we have*

$$\gamma - \delta = 1. \tag{3.14}$$

Proof. Equation (3.14) could be proved in much the same way as equation (3.1)*, but it is much easier to derive it from (3.1), which we are just going to do. Let a point move along the circuit of a polygon in some definite sense beginning the movement from the lowest vertex v. Let it trace the whole circuit once and return to the vertex v. The point will go up and down several times. It is clear that the number of ups is equal to the number of downs. On the other hand, every up begins at the vertex protruding or entering downwards and every down begins at the vertex protruding or entering upwards. The number of ups therefore is $\gamma + \beta$ and the number of downs is $\delta + \alpha$. Hence $\gamma + \beta = \delta + \alpha$, which together with (3.1) yields (3.14), thus proving Lemma 3.

Proof of Theorem 2. We shall assume that the boundary of every hole has no points in common either with the circuit of the polygon M or with the boundaries of other holes. As in the proof of Theorem 1 we number all the vertices in increasing order of their heights and represent the sum by the Euler characteristic $\chi(M) = V - E + F$ on the vertices as follows:

$$V - E + F = (1 - E_1 + F_1) + \dots + (1 - E_V + F_V). \tag{3.15}$$

As before, two facts can be easily verified: first, if a vertex v_i is an interior point of the polygon M, then the corresponding term $1 - E_i + F_i = 0$; second, the sum

---

* It will be noticed that if we change upwards and downwards, then all vertices protruding upwards will become protruding downwards, etc. Thus the proof of (3.14) can immediately be obtained from Lemma 2.

of all the terms on the right of (3.15) which corresponds to the partition vertices lying on the external circuit of M is 1.

Now consider some hole C and all the vertices on its boundary. As noticed, C together with its boundary is a simple polygon. Let a point v_i considered to be a vertex of the polygon C protrude downwards. Being a vertex of M, this point then *enters downwards*. Therefore, as we have seen in the proof of Theorem 1, $1 - E_i + F_i = -1$. If the vertex v_i of the polygon C *enters upwards*, then, being a vertex of the polygon M, it protrudes upwards and hence $1 - E_i + F_i = 1$. For all the other vertices of the polygon C we have $1 - E_i + F_i = 0$. Thus the sum of all the terms on the right of (3.15) which correspond to the hole C is $\delta - \gamma$ and by Lemma 3 $\delta - \gamma = -1$. Repeating this reasoning for every hole separately we obtain the equation $\chi(M) = 1 - n$. So we have proved the special case of Theorem 2.

Let a figure A be the union of a finite number of polygons having pairwise no points in common. These polygons are called *components* of the figure A; we denote their number by $c(A)$. In what follows, let $c^*(A)$ denote the number of the *components of a complement of A with respect to the plane.* One of the components is unbounded, the others being holes relating to one polygon or another. Theorem 2 immediately yields the following

Corollary. *The Euler characteristic of the figure A is*

$$\chi(A) = c(A) - c^*(A) + 1. \qquad (3.16)$$

Problem

10. Prove that for any partition of a polygon having n holes the sum $V + E + F + n$ is odd.

4. The Euler Characteristic and the Sum of the Exterior Angles of a Polygon

In this section it is shown that the Euler characteristic for a polygon can be easily expressed in terms of the sum of its exterior angles. Thus in particular another proof of Theorem 1 is given here. As before, we begin with the case of simple polygons.

Let M be a simple polygon. To orient a simple polygon is to indicate which of the two possible directions of tracing its circuit is considered to be positive. It is usually the direction for which the interior points of the polygon remain at the left (Fig. 19). The opposite direction is then negative. We may also say that the tracing of a circuit in the positive direction is traversing that circuit

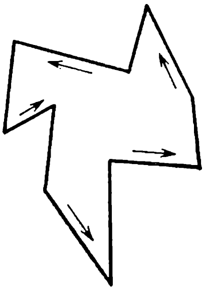


Fig. 19

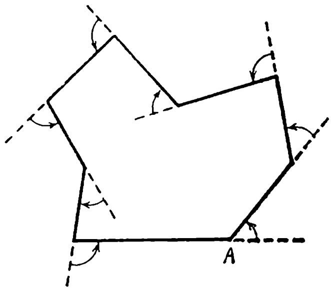


Fig. 20

in counterclockwise fashion. This is due to the above convention about the positive direction in measuring angles.

Suppose M is a simple oriented polygon and we move along its circuit in the positive direction. An *exterior angle* A of a polygon is the angle between the side of the polygon (in the positive direction) with an endpoint at this vertex and the other side extended through the vertex A (Fig. 20). It is natural to consider an exterior angle to be the measure that describes a rotation from one side of the angle to the other through the exterior of the polygon in the positive direction. If, for example, an interior angle whose vertex A is close to π, then the exterior angle is close to zero and the circuit is only slightly rotated. It is easy to verify that in general the interior angle φ and the exterior angle ψ at vertex A are related thus

$$\varphi + \psi = \pi \tag{4.1}$$

(taking into account the sign of the angle ψ). Notice that interior angles of a polygon are always considered to be positive. From (4.1) it follows, in particular, that an exterior angle of a polygon is positive if and only if the corresponding interior angle is less than π.

Lemma 4. *Let M be a simple polygon having n sides. Then the sum Φ of all its interior angles is $(n-2)\pi$ and the sum Ψ of all its exterior angles is 2π regardless of the number of sides.*

The reader is already familiar with the special case of the lemma which refers to convex polygons.

We prove the first statement of the lemma by induction on the number of sides of the polygon. For triangles, it is known. Assuming it to be true for every polygon with the number of sides less than n we prove it for a polygon with n sides. We draw a diagonal in the polygon M to divide it into two simple polygons M_1 and M_2 (see proof of Lemma 2). Let Φ_1 and n_1 denote the sum of the interior angles and the number of sides of M_1, and Φ_2 and n_2 the same quantities for M_2. Under the induction hypothesis we have $\Phi_1 = (n_1 - 2)\pi$, $\Phi_2 = (n_2 - 2)\pi$. Besides it is obvious that $\Phi = \Phi_1 + \Phi_2$ and $n_1 + n_2 = n + 2$. Therefore

$$\Phi = (n_1 - 2)\pi + (n_2 - 2)\pi = (n_1 + n_2 - 4)\pi$$
$$= (n-2)\pi,$$

proving the first statement of the lemma.

Let φ_i be an interior angle of the polygon and ψ_i be the corresponding exterior angle, $i = 1, \ldots, n$. Then $\varphi_i + \psi_i = \pi$. Therefore

$$\Psi = \sum_{i=1}^{n} \psi_i = \sum_{i=1}^{n} (\pi - \varphi_i) = n\pi - \sum_{i=1}^{n} \varphi_i = n\pi - n\pi + 2\pi = 2\pi.$$

Lemma 4 is proved.

Let us now prove the *formula*

$$\Psi = 2\pi (V - E + F) \tag{4.2}$$

relating the sum of exterior angles of a simple polygon to its Euler characteristic, the notation being that of Lemma 4.

Proof. Let M be a simple polygon decomposed into faces, α be any interior angle of an arbitrary face, $\sum\alpha$ be the sum of all such angles. Then $\sum\alpha = \sum_1\alpha + \sum_2\alpha$, where $\sum_1$ is the sum of all those angles α whose vertices lie on the boundary of the polygon M and $\sum_2$ is the sum of the remaining angles, i.e. such angles whose vertices lie inside M.

Let V_1 be the number of vertices lying on the boundary of M and V_2 the number of interior vertices. Then $V = V_1 + V_2$. The sum of all the angles α at each interior vertex is 2π, therefore $\sum_2 = 2V_2\pi$. From this, taking into account (4.1), we obtain the following expression for the sum of exterior angles of the polygon M:

$$\Psi = \sum_{i=1}^{V_1} \psi_i = \sum_{i=1}^{V_1} (\pi - \varphi_i) = V_1\pi - \sum_{i=1}^{V_1} \varphi_i = V_1\pi - \sum{}_1\alpha$$
$$= V_1\pi - \sum \alpha + \sum{}_2\alpha = (V_1 + 2V_2)\pi - \sum \alpha. \tag{4.3}$$

Let m be the number of sides (or angles) of that face where this number is the greatest. By Lemma 4

$$\sum \alpha = [F_3 + 2F_4 + \ldots + (m-2)F_m]\pi, \tag{4.4}$$

where F_3 is the number of triangular faces, F_4 is the number of quadrangular faces, . . ., F_m is the number of m-gonal faces. Equations (4.3) and (4.4) yield

$$\Psi = [V_1 + 2V_2 - F_3 - 2F_4 - \ldots - (m-2)F_m]\pi. \tag{4.5}$$

Let E_1 be the number of edges lying on the boundary of the polygon M and E_2 be the number of the interior edges. Then $E = E_1 + E_2$. Since each interior edge is shared by two faces and each boundary edge belongs to one face, by summing edges over all faces, we get

$$3F_3 + 4F_4 + \ldots + mF_m = E_1 + 2E_2. \tag{4.6}$$

The obvious equation

$$F = F_3 + F_4 + \ldots + F_m$$

yields

$$F_3 + 2F_4 + \ldots + (m-2)F_m$$
$$= (3F_3 + 4F_4 + \ldots + mF_m) - 2(F_3 + F_4 + \ldots + F_m)$$
$$= (3F_3 + 4F_4 + \ldots + mF_m) - 2F. \tag{4.7}$$

From (4.5), (4.7), (4.6), $V_1 = E_1$ and $E = E_1 + E_2$ we get

$$\Psi = [V_1 + 2V_2 - E_1 - 2E_2 + 2F + V_1 - E_1]\pi$$

or

$$\Psi = 2\pi(V - E + F), \tag{4.2}$$

which was to be proved.

It is clear that (4.2) and Lemma 4 again yield the statement of Theorem 1.

Now let M be a polygon with holes. We assume for simplicity that the boundary of each hole, which is a circuit, has no points in common either with the boundaries of the other holes or with the external circuit of M. The orientation of the polygon is given as before, namely, as positive is considered the direction of tracing the boundary of the polygon for which the interior points of M remain at the left. This means that the external circuit is traversed counterclockwise, while the boundary of each hole is traced clockwise (Fig. 21). Also preserved is the definition of an exterior angle. It is easy to verify the following fact. If A is a vertex of M lying on the boundary of the hole of F, ψ is the exterior angle of M at the point A and ω is the exterior angle of a simple polygon F at the same point, then $\psi = -\omega$. In Fig. 21, for example, an exterior angle of the polygon M at the point A is the angle DAB and that of polygon F is EAC. These angles are equal in magnitude as vertical ones. It is clear that they have opposite signs. Thus the sum of exterior angles of a polygon taken over all the vertices of some one of its holes is -2π. Hence, for a polygon M with n holes we have $\Psi = 2\pi(1 - n)$ and, in view of Theorem 2, $\Psi = 2\pi\chi(M)$. The last equation could be proved irrespective of Theorem 2.

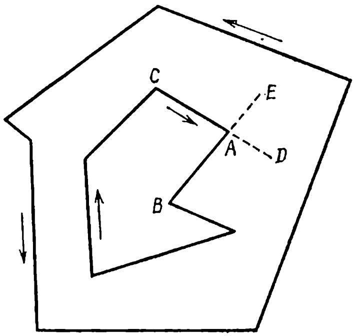


Fig. 21

Problems

11. Let a simple pentagon be decomposed into convex polygonal faces, so that each side of the pentagon is a side of some face. Prove that if the number of faces is not less than 5, then there is an angle $\geqslant 2\pi/5$ at least in one of them.

12. Let a simple polygon M be decomposed into simple polygons $M_1, \ldots, M_n$ so that every two polygons M_i and M_j either have no common points at all or their intersection is a simple nonclosed broken line that lies on the boundary of each of them. These broken lines may degenerate, i.e. may be points. Call the polygons $M_1, \ldots, M_n$ *faces of the partition M*. An *inner vertex of the*

partition is such an interior point of M which belongs to three (or more) faces. A *boundary vertex of a partition* is a point on the boundary of M which belongs to two (or more) faces. *Edges of a partition* are simple nonclosed broken lines that lie on the boundaries of faces and join vertices of the partition. Applying the reasoning used in the proof of (4.2) prove that for such an understanding of a partition the equation $\chi(M) = V - E + F = 1$ is true.

13. Using the results of Problem 12 prove that a complete graph with five vertices cannot be embedded in the plane.

5. Applying the Euler Characteristic to Calculation of Areas..

Suppose horizontal straight lines are drawn in the plane so that the distance between every pair of adjacent lines is equal to 1, vertical lines being drawn in the same way (Fig. 22). Such lines divide the plane into squares with sides equal to 1, which thus have unit areas.

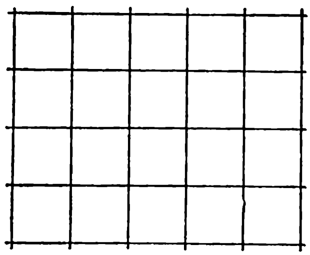


Fig. 22

The set of vertices of all the squares is called a *point lattice* and the vertices themselves the *nodes of the lattice.*

We apply the Euler characteristic to calculate the areas of polygons whose vertices are all at the lattice nodes. Such polygons are called *lattice polygons.* In this section only lattice polygons are considered.

If M is a simple lattice polygon, then for its area $S(M)$ the following formula holds:

$$S(M) = i + \frac{b}{2} - 1, \qquad (5.1)$$

where i is the number of nodes lying in the interior of the polygon M and b is the number of nodes on its boundary. For example, for the polygon M (Fig. 23) we have $i = 13$, $b = 16$, and therefore $S(M) = 13 + \frac{16}{2} - 1 = 20$. Calculation of the area in this case reduces to solving the problem of calculating lattice nodes of two different types. Formula (5.1) was obtained in 1899 by the Austrian mathematician G. Pick (1859-1943 [?]).

The proof of Pick's formula is carried out in three steps.

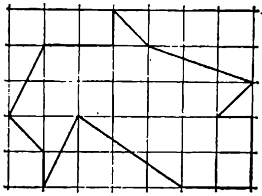


Fig. 23

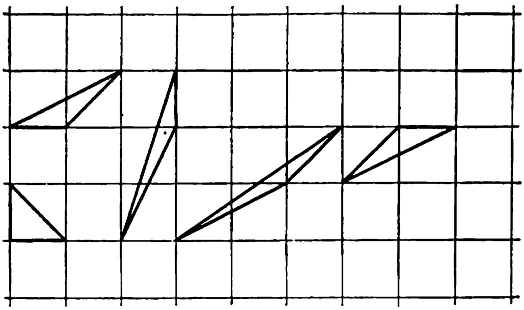


Fig. 24

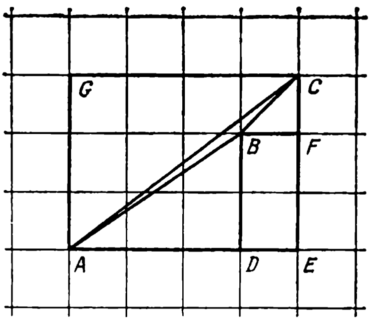


Fig. 25

1. A lattice triangle is said to be *primitive* if there are lattice nodes neither inside it nor on its sides; examples of such triangles are shown in Fig. 24. The first step of the proof consists in verifying that the area of any primitive triangle Δ is equal to 1/2:

$$S(\Delta) = 1/2. \tag{5.2}$$

Let ABC be a primitive triangle (Fig. 25), $S(ABC)$ be its area, and $AGCE$ be the smallest rectangle with ver-

tices at the nodes of the lattice containing the triangle ABC. Let D and F be the feet of the perpendiculars dropped from the point B to the straight lines AE and CE respectively. We introduce the following notation for line segment lengths:

$$|AD| = p, \quad |AE| = q, \quad |EF| = r, \quad |EC| = s.$$

Clearly the numbers p, q, r, and s are integers. We find the areas of the triangles ACE, ABD, BCF and that of the rectangle $BDEF$. We have

$$S(ACE) = \frac{qs}{2}, \qquad S(ABD) = \frac{pr}{2},$$

$$S(BCF) = \frac{(q-p)(s-r)}{2}, \qquad S(BDEF) = (q-p)r.$$

Hence

$$S(ABC) = \frac{qs}{2} - \frac{pr}{2} - \frac{(q-p)(s-r)}{2} - (q-p)r$$

or, after a simplification,

$$S(ABC) = \frac{1}{2}(ps - qr). \tag{5.3}$$

We have not yet used the condition that ABC should be a primitive triangle. We show that if this condition is fulfilled, then in (5.3) we have $ps - qr = 1$ and hence (5.2) is true. We denote by $N(M)$ the number of lattice nodes lying *inside* (but not on the boundary!) of the polygon M and by $N(PQ)$ the number of nodes lying on the segment PQ and different from its ends. Thus, for example, $N(ABC) = N(AB) = N(AC) = N(BC) = 0$ (Fig. 25).

We first find the number $N(AGCE)$. Since $|AE| = |GC| = q$ and $|AG| = |CE| = s$, the rectangle $AGCE$ contains all together $(q+1)(s+1)$ nodes. Of these $2(q+1) + 2(s+1) - 4$ nodes lie on the boundary of the rectangle, the rest lying inside it. Thus $N(AGCE) = (q+1)(s+1) - 2(q+1) - 2(s+1) + 4 = (q-1)(s-1)$. Further, since AC is a diagonal of $AGCE$ dividing it into two triangles and $N(AC) = 0$, we have

$$N(ACE) = \frac{1}{2} N(AGCE) = \frac{1}{2}(q-1)(s-1).$$

Similarly

$$N(ABD)=\frac{1}{2}(p-1)(r-1),$$

$$N(BCF)=\frac{1}{2}(q-p-1)(s-r-1)$$

$$N(BDEF)=(q-p-1)(r-1),$$

$$N(BD)=r-1, \quad N(BF)=q-p-1.$$

It is clear from Fig. 25 that

$$N(ACE) = N(ABC) + N(ABD) + N(BCF) + N(BDEF) + N(AB) + N(BC) + N(BD) + N(BF) + 1,$$

the unity at the right corresponding to the point B. Substituting in this formula the obtained values of numbers $N(M)$ for different polygons M we get

$$\frac{1}{2}(q-1)(s-1)=\frac{1}{2}(p-1)(r-1)+\frac{1}{2}(q-p-1)\times(s-r-1)+(q-p-1)(r-1)+(r-1)+(q-p-1)+1.$$

After a simplification this yields $ps - qr = 1$, proving (5.2).

2. Let M be a simple lattice polygon. We show that it can be divided into primitive triangles. Since a polygon is always assumed to be given as a partition into convex polygons and each convex lattice polygon is divided into lattice triangles, it remains to show that every lattice triangle can be divided into primitive triangles. Suppose there are lattice nodes inside $\triangle$ or on its boundary. Join some inner node to all the vertices of the triangle $\triangle$ or join some node lying on a side of $\triangle$ to its opposite vertex. The resulting partition of $\triangle$ into two or three triangles is such that each of them has fewer nodes inside itself or on its sides than $\triangle$ has. We then apply the same construction to those of the triangles obtained which are not primitive. It is clear that after a finite number of steps we arrive at a partition of $\triangle$ into primitive triangles.

3. We show that for any partitioning of a simple lattice polygon M into primitive triangles their number is

equal to $2i + b - 2$, where i is the number of interior nodes and b is the number of boundary nodes respectively. From this and from (5.2) we obtain Pick's formula. Let V, E, and F denote the numbers of vertices, edges and (triangular) faces of a partition. Since the vertices of a partition coincide with the nodes lying inside M, we have $V = i + b$. Besides, $E = E_i + E_b$, where E_i is the number of the interior edges and E_b is the number of boundary edges of the partition, and

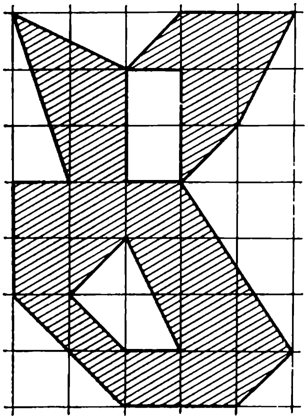


Fig. 26

$$E_b = b \qquad (5.4)$$

and

$$3F = 2E_i + E_b. \qquad (5.5)$$

Equation (5.5) is obtained by summing the edges over all (triangular) faces considering that every boundary edge belongs to one face and every interior edge to exactly two faces. Equation (5.5) can now be written as follows:

$$3F = 2(E - E_b) + E_b = 2(E - b) + b,$$

whence $E = \frac{1}{2}(3F + b)$. Substituting the obtained values of V and E in Euler's formula

$$V - E + F = 1, \qquad (5.6)$$

we get $i + b - \frac{1}{2}(3F + b) + F = 1$ or $F = 2i + b - 2$, which was to be proved. Pick's formula (5.1) is thus proved.

It is interesting to note that there is an analogue of Pick's formula (5.1) for lattice polygons with holes (Fig. 26). It is of the form

$$S(M) = i + \frac{b}{2} - \chi(M) + \frac{1}{2}\chi(\partial M), \qquad (5.7)$$

where ∂M is the boundary of a polygon M. It is clear that (5.7) generalizes (5.1), since for a simple polygon we have

$$\chi(M) = 1 \text{ and } \chi(\partial M) = 0.$$

The proof of (5.7) is the same as that of (5.1), differing only at the very end. Namely, instead of (5.4) we now have, by the definition of the Euler characteristic for a boundary,

$$b - E_b = \chi(\partial M) \tag{5.8}$$

and instead of (5.6) we have

$$V - E + F = \chi(M). \tag{5.9}$$

Therefore (5.5), after a transformation with an account of (5.8), becomes

$$3F = 2E - b + \chi(\partial M)$$

or

$$E = \frac{1}{2}[3F + b - \chi(\partial M)].$$

Substituting the obtained value of E and $V = i + b$ in (5.9) we get

$$i + b - \frac{1}{2}[3F + b - \chi(\partial M)] + F = \chi(M).$$

From this we find the number F:

$$F = 2i + b - 2\chi(M) + \chi(\partial M).$$

Since all faces of the partition are primitive triangles having an area of 1/2, formula (5.7) is proved.

For other analogues of Pick's formula in the plane and in space see [3].

Problems

14. Prove that for the area $S(M)$ of a simple lattice polygon M the following inequality holds:

$$S(M) \geqslant G - \frac{L}{2} - 1. \tag{5.10}$$

Here G denotes the total number of lattice nodes lying inside M (i.e. $G = i + b$), L denotes the perimeter of the polygon, i.e. the length of its boundary. If the polygon is divided into unit squares with vertices at lattice nodes, then (5.10) becomes an equation.

15. Let there be a point lattice in the plane, described earlier (we call it a 1-lattice). We draw additional horizontal and vertical lines so that a partition of the plane into squares with a side of $\frac{1}{2}$ results. We call the vertices of the squares nodes of a $\frac{1}{2}$ lattice. Let M be a polygon (simple or with holes) whose vertices are at

nodes of a $\frac{1}{2}$ lattice. Let i_2 and b_2 be the numbers of nodes of a $\frac{1}{2}$ lattice respectively inside and on the boundary of the polygon (similar numbers for a 1-lattice are denoted by i_1 and b_1). Prove that the area of the polygon is

$$S(M)=\frac{1}{4}\left[i_2+\frac{b_2}{2}-\chi(M)+\frac{1}{2}\chi(\partial M)\right].$$

If, however, vertices of the polygon are at the nodes of a 1-lattice (and therefore also at the nodes of a $\frac{1}{2}$ lattice), then

$$S(M)=\frac{1}{3}\left[(i_2-i_1)+\frac{1}{2}(b_2-b_1)\right],$$

i.e. the terms containing the Euler characteristic cancel out.

6. Euler's Formula for Space

We proceed to study the Euler characteristic for three-dimensional figures. Partitioning such a figure gives rise not only to vertices, edges and faces, but also to three-dimensional cells (the three-dimensionality of a cell implies that it contains a sphere or, equivalently, it does not lie in any plane). It is now natural to take the Euler characteristic for a figure to be the number

$$\chi(\Phi) = V - E + F - C,$$

where C is the number of its three-dimensional cells.

We start by calculating the Euler characteristic for the (three-dimensional) space R itself. Let there be a finite family of planes $Q_1, \ldots, Q_n$ in the space. The planes divide the space into a finite set of three-dimensional cells; we denote their number by C. Let Q_i be some plane of the family. Its intersection with the other planes yields in Q_i a finite set of straight lines which, consequently, form a partition of that plane. The vertices, edges and faces of the partition of all the given planes are called respectively the vertices, edges and faces of the partition of the space. It may happen that the partition has neither vertices nor edges; it can be easily seen that this happens if and only if all the planes are parallel to one another; in this case it is natural to think of the planes themselves

as faces of the partition. Further, it is not hard to show that a partition has no vertices if and only if all the given planes are parallel to some straight line L in space (Fig. 27) (check it!). Let Q be a plane perpendicular to such a straight line L. Then the numbers of three-dimensional cells, faces and edges of the space partition under consideration are respectively equal to those of the faces, edges and vertices of the partition of Q resulting from the intersection of Q with the planes of the family. It follows that in this special case we have $V - E + F - C = -1$. It turns out that this equation always holds, i.e. the Euler characteristic for space is equal to -1 (see (6.1)).

The straight lines of the intersection of planes of a given family will be called *lines of partition*. To calculate the Euler characteristic we shall need the concept of the *multiplicity* of a vertex (and of a line of the partition). They are defined just in the same way, i.e. as the number of planes of the family that pass through that vertex (or line).

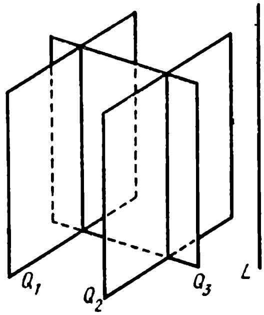


Fig. 27

Lemma 5. *Given planes $Q_1, \ldots, Q_m$ and straight lines $L_1, \ldots, L_n$ in space, it is possible to draw a new plane Q which is not parallel to any one of the given planes and to any one of the given straight lines.*

Proof. We take in Q_1 some point O and draw through it planes $Q'_2 \parallel Q_2, \ldots, Q'_m \parallel Q_m$ and straight lines $L'_1 \parallel L_1, \ldots, L'_n \parallel L_n$. Let φ_i be the angle between the planes Q_1 and Q'_i $(i = 2, \ldots, m)$ and ψ_j be the angle between Q_1 and L'_j $(j = 1, \ldots, n)$. As usual, we take as a measure of the angle between two planes the value of the corresponding plane angle, with $0 \leqslant \varphi_i \leqslant \frac{\pi}{2}$ and $0 \leqslant \psi_j \leqslant \frac{\pi}{2}$ for all i and j. If the angles $\varphi_2, \ldots, \varphi_m, \psi_1, \ldots, \psi_n$ are all zero, then any plane not parallel to Q_1 will be a plane satisfying our requirement. Otherwise, we choose the smallest positive angle among these angles. Let it be ψ_1, for example. We draw through the point O in Q_1 a straight line L different both from all the

straight lines $L'_1, \ldots, L'_n$ and from the lines of intersection of Q_1 with each of the planes $Q'_2, \ldots, Q'_m$ (see the proof of Lemma 1). We draw through L a plane Q which makes with Q_1 a positive angle less than ψ_1. Clearly Q does not coincide with any one of the planes $Q'_2, \ldots, Q'_m$ and contains none of the straight lines $L'_1, \ldots, L'_n$, and is therefore the required plane, proving the lemma.

We prove Euler's formula for the space R, i.e. we verify that for any partition of it by a finite number n of planes we have

$$\chi(R) = V - E + F - C = -1. \tag{6.1}$$

We assume that the partition $\mathscr{D}_R$ under consideration has vertices and hence edges. The proof of (6.1) for the case with no vertices will be left to the reader.

We prove (6.1) by the method of a moving plane. We first draw auxiliary straight lines through each pair of vertices not lying on the same line of partition. For example, if the space is divided by the six planes obtained by extending all faces of the cube, 16 auxiliary straight lines will be drawn through 4 pairs of opposite vertices of the cube itself and through each pair of opposite vertices of each face. Using Lemma 5 we draw a plane Q which is not parallel to any one of the given planes or to any one partition line or to any one auxiliary straight line. It is Q that will be the moving plane. We make two assumptions about it: we assume that it is, first, horizontal (this can be achieved by rotating the whole space in the required manner) and, second, in its initial position it is below all the vertices. It follows from the first assumption, in particular, that the vertices of the partition are all at different heights in space. We number them in increasing order of height, i.e. let v_1 be the lowest vertex, v_2 the next in height and finally v_V the uppermost vertex.

To prove (6.1) we can, for example, express the numbers E, F, and C in terms of V and n. By analogy with the proof of (1.2) we again need additional information, and even a much greater amount of it than in the plane case. Namely, it will be necessary to take into account the multiplicities of all the vertices, the total number m of lines of partition and their multiplicities, and also the

number of the lines of partition passing through each vertex. For simplicity we carry out such a proof only for the special case where each vertex has multiplicity 3 and each line of partition has multiplicity 2. However, the reasoning below can be applied to the general case as well (which will be done in what follows) and the difference between the special and the general case shows itself only in the formulas for E, F, and C.

Consider a moving plane Q in its initial position. When Q intersects the planes of the family a family of n straight lines results which form a partition $\mathscr{D}_Q$ of Q. Each line of partition $\mathscr{D}_R$ of space has a corresponding vertex in the partition $\mathscr{D}_Q$ of the plane, namely, the point of intersection of that line and Q. Similarly, the faces and three-dimensional cells of $\mathscr{D}_R$ intersected by Q have corresponding edges and faces in $\mathscr{D}_Q$. This partition of Q has thus m vertices, each of multiplicity 2. According to (1.3) and (1.4) $\mathscr{D}_Q$ has respectively $n + 2m$ edges and $1 + n + m$ faces. In its initial position therefore Q intersects

$$\begin{aligned} m &\text{ edges,} \\ n + 2m &\text{ faces, and} \\ 1 + n + m &\text{ three-dimensional cells} \end{aligned}$$

of $\mathscr{D}_Q$.

Now let the plane Q move up and keep its horizontal position. It follows from the first assumption about Q and from Lemma 5 that each edge of $\mathscr{D}_R$ (and, similarly, each face and each three-dimensional cell of it), except those intersected by Q in its initial position, has a single lowest vertex. Two conclusions follow. First, *new cells* of $\mathscr{D}_R$ appear only at the moment when Q passes through the vertices of $\mathscr{D}_R$. Second, the number of new edges, faces and three-dimensional cells resulting when Q passes through the vertex v_i is equal respectively to the number of vertices, *bounded edges* and *bounded faces* of the partition of Q which is formed by the straight lines of its intersection with the planes of the family passing through the point v_i at the moment when Q is just above v_i (the new cells of $\mathscr{D}_R$ are "born" as it were from the vertex v_i) (see Fig. 28). By virtue of the assumption about the multiplicities of vertices and lines, the plane Q is partitioned by three straight lines in general position (see Problem 3). Consequently, three new edges, three faces and one three-dimensional cell appear. This happens

every time the plane passes through a vertex. Therefore

$$E = m + 3V, \quad F = n + 2m + 3V, \quad C = 1 + n + m + V. \quad (6.2)$$

Hence

$$\chi(R) = V - E + F - C = V - m - 3V + n + 2m$$
$$+ 3V - 1 - m - n - V = -1.$$

Thus Euler's formula (6.1) has been proved on the assumption that each vertex of the partition has multiplicity 3 and that each line of partition has multiplicity 2.

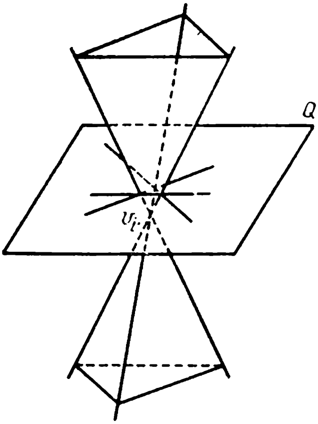


Fig. 28

In the general case we shall not seek for formulas expressing E, F, and C in terms of V and n and other data but use a device already employed earlier for the plane. Namely, we shall take the alternating sum $V - E + F - C$ over the vertices, i.e. we represent it as

$$\begin{aligned}\chi(R) &= V - E + F - C \\ &= (-E_0 + F_0 - C_0) \\ &\quad + (1 - E_1 + F_1 - C_1) \\ &\quad + (1 - E_2 + F_2 - C_2) \\ &\quad + \ldots + (1 - E_V + F_V - C_V),\end{aligned} \quad (6.3)$$

where E_0, F_0, and C_0 are the numbers of cells in the partition of the space the plane Q meets in its initial position, E_1, F_1, and C_1 are the numbers of the cells appearing after Q passes through the first vertex, and so on. As already pointed out, the sum

$$S_0 = E_0 - F_0 + C_0$$

is equal to the Euler characteristic of the plane and each sum

$$S_i = E_i - F_i + C_i \quad (i = 1, \ldots, V)$$

is equal to the Euler characteristic for the figure made up of bounded cells of the partition of Q generated by the straight lines of its intersection with those planes of the family passing through the vertex v_i. By the statement of Problem 7 we have $S_i = 1$. Substituting the obtained values of the sum S_i in (6.3) we get $\chi(R) = -1$.

7. Euler's Formula for Convex Polytopes and Its Consequences

A *convex polytope* is an intersection of a finite number of half-spaces provided that it is, first, *bounded*, i.e. contained in some sphere, and, second, *three-dimensional*, i.e. contains some other sphere, or equivalently, it does not lie in any plane. It is also assumed in the definition of a convex polytope that each half-space contains a bounding plane.

All the points of a convex polytope are divided into interior and boundary points. A point of a polytope X is said to be *interior* if there is a sphere which lies entirely in X with centre at that point. A point is said to be *boundary* if any sphere with centre at that point contains both the points of X and the points of its complement with respect to the space. All boundary points form the *boundary* of a polytope. The boundary consists of a finite number of *faces* which are convex polygons. The sides and vertices of such polygons are called *edges* and *vertices* of the polytope, respectively.

We now prove famous *Euler's formula*

$$V - E + F = 2 \tag{7.1}$$

which is true for any convex polytope.

Proof. Assume, as usual, that the vertices of the polytope are all at different heights and are numbered in such an order $v_1, v_2, \ldots, v_V$ that each subsequent vertex is above the preceding one. Denote by F_i $(i = 1, \ldots, V - 1)$ the number of the polytope's faces for which the point v_i is the lowest vertex or, in other words, which "go upwards" from the vertex v_i. Let E_i $(i = 1, \ldots, V - 1)$ be the number of the polytope's edges with the vertex v_i as their lower endpoint (or "go upwards" from this vertex). Clearly, there are no such faces and edges for the uppermost vertex v_V. Since an equal number of edges and faces adjoin v_i and they all "go upwards" from it, we have

$$E_1 = F_1. \tag{7.2}$$

Now cut the polytope X by a horizontal plane Q_i which is just above the vertex v_i $(i = 1, \ldots, V - 1)$. Its section is a convex polygon M_i. Each of the edges E_i of the polytope going upwards from the point v_i has a

corresponding vertex in M_i. Similarly, each of the F_i faces going upwards from the same point has a corresponding side in M_i. The vertices and sides of the polygon form a simple (nonclosed) broken line (possibly,

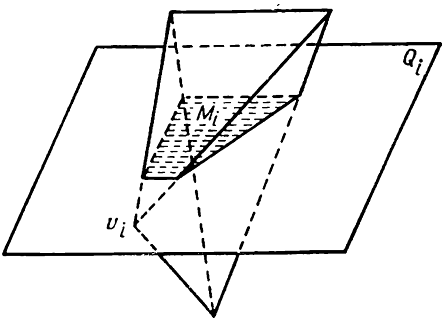


Fig. 29

consisting of just one vertex). Since the Euler characteristic for such a broken line is equal to unity, we have

$$E_i - F_i = 1 \quad (i = 2, \ldots, V - 1) \qquad (7.3)$$

(Fig. 29 shows one of the simplest cases: X is a tetrahedron, the broken line consists of one line segment). The total number of edges of the polytope is $E = E_1 + \ldots + E_{V-1}$ and the total number of its faces is $F = F_1 + \ldots + F_{V-1}$. Therefore using (7.2) and (7.3) we get

$$\begin{aligned} V - E + F &= V - (E_1 + \ldots + E_{V-1}) + (F_1 + \ldots + F_{V-1}) \\ &= V - (E_1 + \ldots + E_{V-1}) + E_1 + (E_2 - 1) \\ &\quad + \ldots + (E_{V-1} - 1) = V - (E_1 + \ldots + E_{V-1}) \\ &\quad + (E_1 + \ldots + E_{V-1}) - (V - 2) = 2. \end{aligned}$$

Euler's formula (7.1) is thus proved.

Notice that this formula expresses the property of the *boundary* ∂X of the polytope. The above argument does not yet prove, however, that the Euler characteristic for the boundary is equal to 2: as a matter of fact, we took into account only the "natural"* partition of the boundary into cells, whereas in order to prove the equation

$$\chi(\partial X) = 2 \qquad (7.4)$$

we must show that (7.1) is true for any partition.

* That is, its "actual" vertices, edges, and faces.

So, suppose that in addition to the natural partition there is another ("new") partition of the boundary ∂X into convex cells of dimensions from 0 to 2. We shall follow the second point of view on the two partitions, i.e. we shall assume that all cells are open and that therefore different cells of the same partition have no points in common. Let v, e, and f be the numbers of vertices, edges, and faces of the new partition. We must prove the equation $v - e + f = 2$. Since all the cells are convex, every new cell is in exactly one old cell. Furthermore, all new cells contained in the same old cell make up its partition. Therefore if F is an old open two-dimensional cell, i.e. if it is a face of the polytope without its boundary, and if v_1, e_1, and f_1 are the numbers of new cells of different dimensions that are contained in F, then

$$v_1 - e_1 + f_1 = \chi(F) = 1. \tag{7.5}$$

Similarly, if E is an old open edge, i.e. if it is a natural edge of the polytope without its endpoints, and if v_2 and e_2 are the numbers of new cells contained in E, then

$$v_2 - e_2 = \chi(E) = -1. \tag{7.6}$$

Since all the new cells are contained in ∂X, the sum $v - e + f$ can be taken over the old cells and from (7.5) and (7.6) we therefore get

$$\chi(\partial X) = v - e + f = V - E + F = 2.$$

Thus (7.4) is proved.

Incidentally, the last argument shows that the Euler characteristic for the boundary of a polytope is equal to the sum of the Euler characteristics for its open cells. This property is called *additivity* and is considered below in a more general form.

Before deriving consequences from Euler's formula (7.1) we first establish some simple but useful relations. Notice that throughout this section, by a polytope we shall always mean a convex polytope.

The *degree* of a vertex of a polytope is the number of edges emanating from it. Clearly the degree of every vertex is at least three. Denote by V_3 the number of vertices of degree 3, by V_4 the number of vertices of degree 4, and so on. Then

$$V = V_3 + V_4 + \ldots + V_m = \sum_{i=1}^{m} V_i. \tag{7.7}$$

Here m is the maximum degree of a vertex and $V_1 = V_2 = 0$. The exact value of the number m will often be of no interest to us, therefore we write (7.7) in shorter form:

$$V = V_3 + V_4 + \ldots = \sum_{i \geqslant 3} V_i. \tag{7.7'}$$

Each face of a polytope is a convex polygon the number of whose sides (or angles) is 3 or 4, and so on. Denote by F_i the number of i-gonal faces of the polytope ($i = 3, 4, \ldots$). Then

$$F = F_3 + F_4 + \ldots = \sum_{i \geqslant 3} F_i. \tag{7.8}$$

Summing the edges over all the vertices and considering that every edge joins two vertices, i.e. it is counted twice, we get

$$2E = 3V_3 + 4V_4 + 5V_5 + \ldots = \sum_{i \geqslant 3} iV_i. \tag{7.9}$$

Similarly, summing the edges over all the faces and considering that every edge belongs to the boundaries of two faces and is, consequently, counted twice, we have

$$2E = 3F_3 + 4F_4 + 5F_5 + \ldots = \sum_{i \geqslant 3} iF_i. \tag{7.10}$$

Since every vertex is at least of degree three, or equivalently, at least three faces emanate from every vertex, we have

$$3F \leqslant 3V_3 + 4V_4 + 5V_5 + \ldots = \sum_{i \geqslant 3} iV_i. \tag{7.11}$$

Since every face has at least three vertices, we have

$$3V \leqslant 3F_3 + 4F_4 + 5F_5 + \ldots = \sum_{i \geqslant 3} iF_i. \tag{7.12}$$

Note that the pair of numbers V and F (and also the pairs of numbers V_3 and F_3, V_4 and F_4, and so on) appear symmetrically in all relations from (7.7) to (7.12), as well as in Euler's formula (7.1), i.e. these relations remain true if the number V is replaced by the number F, the number V_3 by the number F_3, etc., and vice versa. Therefore to any statement, for example, to the one about the polytope faces, deduced from formula (7.1) and relations (7.7) to (7.12) there corresponds a similar (dual) statement about the polytope vertices. This is the so-called *principle of duality*. It may be stated in particular that dual to each other are, for example, equations (7.9) and (7.10), as well as inequalities (7.11) and (7.12). Euler's formula is self-dual.

We proceed to derive consequences from relations (7.1) and (7.7)-(7.12).

From inequality (7.12) and equation (7.10) we get $V \leqslant \frac{1}{3} \sum_{i \geqslant 3} iF_i$ and $E = \frac{1}{2} \sum_{i \geqslant 3} iF_i$. Substituting into (7.1) we have

$$\frac{1}{3} \sum_{i \geqslant 3} iF_i - \frac{1}{2} \sum_{i \geqslant 3} iF_i + \sum_{i \geqslant 3} F_i \geqslant 2$$

or

$$6 \sum_{i \geqslant 3} F_i - \sum_{i \geqslant 3} iF_i \geqslant 12.$$

We separate out in both sums the terms containing F_3, F_4, and F_5. Then

$$6(F_3 + F_4 + F_5) + 6 \sum_{i \geqslant 6} F_i - (3F_3 + 4F_4 + 5F_5) - \sum_{i \geqslant 6} iF_i \geqslant 12$$

or

$$3F_3 + 2F_4 + F_5 \geqslant 12 + \sum_{i \geqslant 6} (i-6) F_i.$$

Since the sum on the right of this inequality is not negative we have

$$3F_3 + 2F_4 + F_5 \geqslant 12. \qquad (7.13)$$

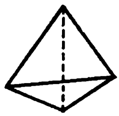
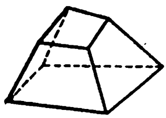
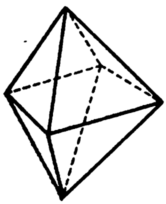
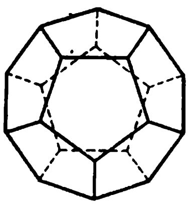
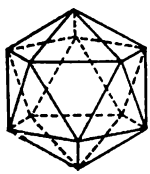
Inequality (7.13) has interesting geometrical consequences. It shows that a convex polytope has necessarily either triangular or quadrangular or pentagonal faces. In particular, there is no convex polytope which would have only hexagonal faces. Assuming $F_4 = F_5 = 0$, (7.13) yields $F_3 \geqslant 4$, and this inequality is exact, i.e. there is a polytope in which $F_4 = F_5 = 0$ and $F_3 = 4$; it is a tetrahedron (a triangular pyramid). If $F_3 = F_5 = 0$, then $F_4 \geqslant 6$. This inequality is also exact, as can be seen by verifying it for a cube. If $F_3 = F_4 = 0$, then $F_5 \geqslant 12$. A dodecahedron demonstrates the exactness of this inequality (see Table 3 below).

The inequality dual to (7.13) is of the following form:

$$3V_3 + 2V_4 + V_5 \geqslant 12. \qquad (7.14)$$

The reader will certainly prove it by himself. From (7.14) we find, in particular, that there is no convex polytope whose vertices would all be of degree 6, and also obtain

TABLE 3

Polytopes		m	n	V	E	F
Tetrahedron		3	3	4	6	4
Hexahedron		4	3	8	12	6
Octahedron		3	4	6	12	8
Dodecahed-ron		5	3	20	30	12
Icosahedron		3	5	12	30	20

the following statements:

$$\text{if } V_4 = V_5 = 0, \quad \text{then } V_3 \geqslant 4,$$
$$\text{if } V_3 = V_5 = 0, \quad \text{then } V_4 \geqslant 6,$$
$$\text{if } V_3 = V_4 = 0, \quad \text{then } V_5 \geqslant 12.$$

The last three inequalities are exact, as can be seen from Table 3.

A convex polytope is said to be *combinatorially regular* if all its faces have the same number (say, m) of sides and all its vertices have the same degree (say, n). Thus, this definition does not require that faces should be equal regular polygons or that polyhedral angles should be equal. This distinguishes a combinatorially regular polytope from a *metrically regular* polytope known from school geometry. (But of course a metrically regular polytope is at the same time combinatorially regular.)

We shall say that a (combinatorially regular) polytope is of type (m, n) if each of its faces is an m-gon and each vertex is of degree n.

We prove that only five different types of combinatorially regular polytopes are possible. As we already know, in a regular polytope each of the numbers m and n may equal 3 or 4 or 5. Out of these numbers we can arrange nine different pairs (m, n). It only remains to check which of the nine pairs can in fact be realized.

So, suppose we have a regular polytope of the type (m, n). Then, $F = F_m$ and, because of (7.10), $2E = mF$. Similarly, $V = V_n$ and, in view of (7.9), $2E = nV$. Solving the system of equations $V - E + F = 2$, $2E = mF$, $2E = nV$ for the numbers V, E and F we get

$$V = \frac{4m}{2m+2n-mn},$$
$$E = \frac{2mn}{2m+2n-mn},$$
$$F = \frac{4n}{2m+2n-mn}.$$

Since these numbers are positive we have

$$2m + 2n - mn > 0$$

or

$$(m-2)(n-2) < 4. \tag{7.15}$$

It is now clear that of all the nine pairs of numbers (m, n) only the following five satisfy inequality (7.15): (3, 3), (4, 3) (3, 4), (5, 3), and (3, 5). Combinatorially regular polytopes corresponding to such pairs do in fact exist; they are: a tetrahedron, hexahedron, octahedron, dodecahedron, and icosahedron. Presented in Table 3 are the values of the numbers m, n, V, E, and F for these polytopes.

We add the following remark to this section for the readers acquainted with an n-dimensional space. If there is a convex polytope in that space which is not in any $(n-1)$-dimensional hyperplane, then its boundary is partitioned into cells of dimensions from 0 to $n-1$. Let α_i denote the number of cells of dimension i $(i = 0, 1, \ldots, n-1)$. Then there is the following analogue of Euler's formula (7.1):

$$\alpha_0-\alpha_1+\alpha_2-\ldots+(-1)^{n-1}\alpha_{n-1}=1+(-1)^{n-1}.$$

Problems

16. Prove that for any convex polytope the number $V + E + F$ is even.

17. Prove Euler's formula (7.1) by the moving plane method without assuming that the vertices of a polytope are at different heights.

18. Prove that if every pair of vertices of a convex polytope is joined by an edge, then the polytope itself is a tetrahedron.

Notice that the convexity requirement in this problem cannot be discarded, as is shown by the example of a solid consisting of three tetrahedrons, $ADCF$, $ADBE$ and $BECF$, each having an edge in common (Fig. 30).

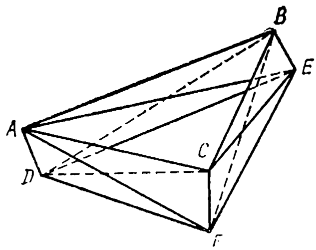


Fig. 30

Another remark. A similar statement does not hold in a multidimensional space. In a four-dimensional space, for example, there are, besides a four-dimensional simplex (the analogue of a tetrahedron), convex polytopes with any number of vertices, at least five, each pair being joined by an edge.

19. Formulate and prove the statement dual to the statement of Problem 18.

20. (The "Cauchy lemma"). Let each edge of a convex polytope be marked with a plus or minus sign. When going round a vertex along the edges a sign reversal may take place (from plus to minus, and vice versa), the number of such reversals clearly being even (possibly, zero). Prove the Cauchy lemma: *there is a vertex of a polytope such that in going round it the number of sign reversals is at most* 2.

Formulate and prove the dual statement.

Notice that this lemma was used by the French mathematician Augustin Louis Cauchy (1789-1857) in 1813 to prove the theorem on the rigidity of boundaries of convex polytopes (for more details see [5]).

21. A convex polytope has five faces. What may the number of its vertices and edges be? Formulate and solve the dual problem.

22. Using equations (7.1) and (7.7) to (7.10) prove the formula

$$\sum_{i\geqslant 3}(2i+2n-ni)\,V_i+2\sum_{i\geqslant 3}(n-i)\,F_i=4n, \qquad (7.16)$$

where n is any natural number. The term F_n is "excluded" from the formula.

Write and prove the dual formula.

23. Using (7.16) for $n = 7$, prove the following theorem. *Let every vertex of a convex polytope be of degree* 3, *and let the polytope have neither triangular nor quadrangular faces. Then it has a pentagonal face which touches another pentagonal or hexagonal face.* (Two faces are said to touch if they have a side in common.)

State and prove the dual theorem.

8. Axioms of the Euler Characteristic

Let us summarize briefly what we have done and outline what we shall do. We have shown that every figure M of some class (say the class of polygons) can be associated with its Euler characteristic $\chi(M)$ by partitioning the figure into cells of different dimensions. In other words, the function χ was defined on a set of figures according to the formula $\chi(M) = V - E + F$ and we were to prove that the function was defined *correctly*, i.e. it does not depend on the way of partitioning the figure. It is usual to call such definitions *constructive*, since they immediately give the rule (the construction) according to which the required function can be calculated.

Now we give a new (*axiomatic*) definition of the Euler characteristic. Namely, we first define some class of figures called elementary. We then define the Euler characteristic as a function χ on that class so that it should satisfy simple and natural requirements (axioms) stated in advance. We shall choose as axioms the properties of the Euler characteristic which are already familiar to the reader. Our main task now will be to prove that such a function χ does indeed exist and is uniquely defined. In addition, we prove that the constructive and axiomatic definitions of the Euler characteristic are equivalent, i.e. they give the same function χ defined on elementary figures.

The new way of defining the Euler characteristic has certain advantages. The main advantage is that it allows us to calculate the value of χ for the entire figure M if we know its values for simpler parts the figure is made up of (convex polygons, as a rule). It is of importance to notice that these parts should not necessarily be faces, edges or cells of the partition.

The axiomatic approach to the definition of the Euler characteristic was proposed in 1955 by the Swiss mathematician H. Hadwiger (1908-1981).

We now proceed to describe (also axiomatically) the set of *elementary figures* $\mathscr{M}$. The set $\mathscr{M}$ of figures in the plane is defined by giving the following two axioms:

(1) every convex polygon C is contained in $\mathscr{M}$ (restated: C is an element of $\mathscr{M}$; in symbols: $C \in \mathscr{M}$);

(2) if figures A and B are contained in $\mathscr{M}$, so are their union $A \cup B$ and intersection $A \cap B$ (in symbols: if $A \in \mathscr{M}$ and $B \in \mathscr{M}$, then $A \cup B \in \mathscr{M}$ and $A \cap B \in \mathscr{M}$).

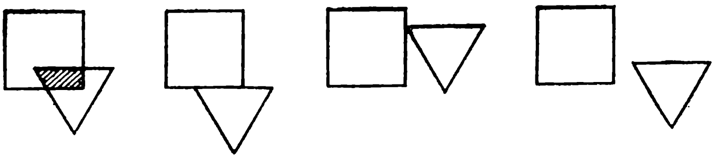


Fig. 31

We derive simple consequences from these axioms. The intersection of two convex polygons may be a convex polygon, a line segment or a point; it may contain no point (i.e. it may be an *empty* figure (Fig. 31)). From now on line segments and points will be called *singular* (convex) polygons. An empty figure is denoted by $\varnothing$. So it follows from the axioms that the class $\mathscr{M}$ of elementary figures must contain all singular polygons and an empty polygon $\varnothing$.

It follows by induction from axiom (2) that the set $\mathscr{M}$ must contain the union and the intersection of any finite number of its elements A_i (in symbols: if $A_i \in \mathscr{M}$ $(i = 1, \ldots, n)$, then $\bigcup_{i=1}^{n} A_i \in \mathscr{M}$ and $\bigcap_{i=1}^{n} A_i \in \mathscr{M}$).

We prove that there is a set of figures in the plane which satisfies axioms (1) and (2). Such is the set consisting of all possible finite unions of convex polygons (pos-

sibly, singular). From now on we shall denote such sets by the letter $\mathscr{M}$ and call their elements *elementary figures* (or simply *figures* for short). Let A and B be figures, i.e. let

$$A = \bigcup_{i=1}^{n} C_i, \qquad B = \bigcup_{j=1}^{m} D_j, \tag{8.1}$$

where C_i and D_j are convex polygons. Clearly axiom (1) holds since the number of n or m components of polygons in (8.1) may be taken equal to 1. The class $\mathscr{M}$ obviously contains the union $A \cup B$ which is of the same form (8.1) as is each of the components A and B. It remains to prove that the intersection $A \cap B$ may be represented as the union of a finite number of convex polygons. This follows immediately from the formula

$$\left(\bigcup_{i=1}^{n} C_i\right) \cap \left(\bigcup_{j=1}^{m} D_j\right) = \bigcup_{i,\, j} (C_i \cap D_j) \tag{8.2}$$

(the "distributive law"), where the indices i and j at the right have the same values as those at the left.

Notice that in addition to the set $\mathscr{M}$ just described there are other classes of figures in the plane which satisfy axioms (1) and (2) but we shall not discuss them.

It is easy to verify that, for example, polygons (not necessarily convex) defined in Sec. 2 are elementary figures. On the other hand, a connected elementary figure equal to the union of nonsingular convex polygons is a polygon. This can be verified in the same way as that a simple hole is a simple polygon (see Sec. 2, p. 20).

Besides polygons, the set $\mathscr{M}$ contains planar graphs, of course. It is not hard to see that any elementary figure can be represented as a union of several polygons and one (possibly, disconnected) planar graph; some of these "components" may be lacking.

By analogy with the class $\mathscr{M}$ of plane elementary figures we define the class $\mathscr{M}$ (L) of elementary figures on the straight line L. Each element of $\mathscr{M}$ (L) is a union of a finite number of line segments; these segments may degenerate into points. It can be easily proved that the set $\mathscr{M}$ (L) satisfies axioms (1) and (2) (this is done in the same way as for the class $\mathscr{M}$).

In a space R we shall only consider the class $\mathscr{M}$ (R) of elementary figures each of which is a union of a finite

number of convex polygons (possibly, singular), the planes of different polygons not necessarily coinciding. In this class, for example, fall the boundaries of convex polygons.

We proceed to define the Euler characteristic. For definiteness we shall speak about the set of figures $\mathcal{M}$. We say that the Euler characteristic χ is given on $\mathcal{M}$ if every elementary figure $A \in \mathcal{M}$ is associated with a number $\chi(A)$, so that the following axioms are satisfied:

(α) *the Euler characteristic for an empty figure is zero,* i.e.

$$\chi(\varnothing) = 0,$$

(β) *for every (including singular) nonempty convex polygon C we have*

$$\chi(C) = 1,$$

(γ) *for any elementary figures A and B*

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Property (γ) is called the *additivity* of the Euler characteristic. Therefore we may say briefly that the Euler characteristic is an additive function of an elementary figure "normalized" by condition (β).

We have already met earlier the additive property of the function χ under more restrictive conditions, though, when polygons A and B had no points in common (see p. 49). The reader is no doubt acquainted with some other additive functions of the polygon: it suffices to say that its area has this property. Indeed, to find the area of the union of two polygons one must take the sum of their areas and subtract from it the area of their intersection, since the latter is accounted twice in the sum.

We derive some consequences from the axioms of the Euler characteristic.

Let n elementary figures $A_1, \ldots, A_n$ be given or, as we shall say, let the n-element part of the whole set $\mathcal{M}$ be given. We write it as follows: $\mathcal{A} = \{A_1, \ldots, A_n\}$. Let $A = \bigcup_{i=1}^{n} A_i$ be the union of all given figures. Then the Euler characteristic of a figure A, if it exists, is expressed in terms of the Euler characteristics for figures $A_1, \ldots,$

A_n as follows:

$$\begin{aligned}\chi(A) = {} & \textstyle\sum^{(1)} \chi(A_i) - \sum^{(2)} \chi(A_{i_1} \cap A_{i_2}) \\ & + \textstyle\sum^{(3)} \chi(A_{i_1} \cap A_{i_2} \cap A_{i_3}) \\ & - \textstyle\sum^{(4)} \chi(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}) \\ & + \ldots + (-1)^{n-2} \textstyle\sum^{(n-1)} \chi(A_{i_1} \cap \ldots \cap A_{i_{n-1}}) \\ & + (-1)^{n-1} \chi(A_1 \cap \ldots \cap A_n). \end{aligned} \tag{8.3}$$

We explain this formula. Here $\sum^{(1)}$ denotes the sum taken over all figures A_i or, equivalently, over all 1-element parts of the set $\mathcal{A}$; $\sum^{(2)}$ denotes the sum taken over all pairs of the figures $\{A_{i_1}, A_{i_2}\}$, where $i_1 \neq i_2$, or, equivalently, the sum is taken over all 2-element parts of the set $\mathcal{A}$; $\sum^{(3)}$ denotes the sum taken over all triples of the figures $\{A_{i1}, A_{i2}, A_{i3}\}$, where the subscripts i_1, i_2, and i_3 assume different values; in other words, this sum is taken over all 3-element parts of the set $\mathcal{A}$ and so on.

If $n = 2$, then (8.3) is nothing than axiom (γ). It should be proved therefore only for $n \geqslant 3$.

Let $n = 3$. Using (γ) we then have

$$\begin{aligned}\chi(A_1 \cup A_2 \cup A_3) = \chi[(A_1 \cup A_2) \cup A_3] = \chi(A_1 \cup A_2) \\ + \chi(A_3) - \chi[(A_1 \cup A_2) \cap A_3]. \end{aligned} \tag{8.4}$$

Using the special case

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

of the distributive law (8.2) we get

$$\begin{aligned}\chi[(A_1 \cup A_2) \cap A_3] & = \chi[(A_1 \cap A_3) \cup (A_2 \cap A_3)] \\ & = \chi(A_1 \cap A_3) + \chi(A_2 \cap A_3) \\ & \quad - \chi(A_1 \cap A_2 \cap A_3). \end{aligned}$$

We substitute this expression for $\chi[(A_1 \cup A_2) \cap A_3]$ in (8.4) and write $\chi(A_1 \cup A_2)$ using axiom (γ). Then we finally have

$$\begin{aligned}\chi(A_1 \cup A_2 \cup A_3) = \chi(A_1) + \chi(A_2) + \chi(A_3) - \chi(A_1 \cap A_2) \\ - \chi(A_1 \cap A_3) - \chi(A_2 \cap A_3) \\ + \chi(A_1 \cap A_2 \cap A_3). \end{aligned} \tag{8.5}$$

For $n = 3$ formula (8.3) is thus proved. We now prove its general case by induction on n, assuming it to be

true for the number of "component" figures equal to $n-1$, i.e. the equation

$$\chi\left(\bigcup_{i=1}^{n-1} A_i\right) = \sum^{(1)} \chi(A_i) - \sum^{(2)} \chi(A_{i_1} \cap A_{i_2}) + \sum^{(3)} \chi(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \ldots + (-1)^{n-3} \sum^{(n-2)} \chi(A_{i_1} \cap \ldots \cap A_{i_{n-2}}) + (-1)^{n-2} \chi(A_1 \cap \ldots \cap A_{n-1}) \tag{8.6}$$

holds. By (γ) we have

$$\chi\left(\bigcup_{i=1}^{n} A_i\right) = \chi\left(\bigcup_{i=1}^{n-1} A_i\right) + \chi(A_n) - \chi\left[\left(\bigcup_{i=1}^{n-1} A_i\right) \cap A_n\right]. \tag{8.7}$$

Using the distributive law in the form

$$\left(\bigcup_{i=1}^{m} A_i\right) \cap B = \bigcup_{i=1}^{m} (A_i \cap B)$$

and the induction hypothesis (8.6) we get

$$\chi\left[\left(\bigcup_{i=1}^{n-1} A_i\right) \cap A_n\right] = \chi\left[\bigcup_{i=1}^{n-1} (A_i \cap A_n)\right] = \sum^{(1)} \chi(A_i \cap A_n) - \sum^{(2)} \chi(A_{i_1} \cap A_{i_2} \cap A_n) + \ldots + (-1)^{n-3} \sum^{(n-2)} \chi(A_{i_1} \cap \ldots \cap A_{i_{n-2}} \cap A_n) + (-1)^{n-2} \chi(A_1 \cap \ldots \cap A_{n-1} \cap A_n), \tag{8.8}$$

where in each of the sums the subscripts i_1, i_2, etc. are all different and run from 1 to $n-1$.

We now substitute (8.6) and (8.8) in (8.7). In (8.6) the sum $\sum^{(1)}$ is taken over all the figures A_i, except for A_n, and besides in (8.7) there is a term $\chi(A_n)$. Combining all these terms we obtain the sum $\sum^{(1)}$ of formula (8.3) being proved, now over all the figures without exception. Further, on the right of (8.6) the sum $\sum^{(2)}$ is taken over all such pairs of figures $\{A_{i_1}, A_{i_2}\}$ that do not contain the figure A_n. To obtain *all* the pairs of figures (with different subscripts) it is necessary to add the pairs

$\{A_1, A_n\}, \ldots, \{A_{n-1}, A_n\}$ to the pairs mentioned. It is just to the latter pairs in (8.8) that the sum $\sum^{(1)}$ corresponds. By adding the sum $\sum^{(1)}$ to the sum $\sum^{(2)}$ from (8.6), from (8.8) we obtain the sum $\sum^{(2)}$ of equation (8.3) being proved, all the terms of this sum having a minus sign. Similarly, the sum $\sum^{(3)}$ of the formula being proved is obtained by adding the sum $\sum^{(3)}$ from (8.6) to $\sum^{(2)}$ from (8.8), and so on. Equation (8.3) is proved.

We write one more special case of this equation which we shall need in what follows:

$$\chi\left(\bigcup_{i=1}^{4} A_i\right)=\sum^{(1)}\chi(A_i)-\sum^{(2)}\chi(A_{i_1}\cap A_{i_2})$$
$$+\sum^{(3)}\chi(A_{i_1}\cap A_{i_2}\cap A_{i_3})$$
$$-\chi(A_1\cap A_2\cap A_3\cap A_4). \tag{8.9}$$

Notice that it is not necessary that the figures $A_1, \ldots, A_n$ in (8.3) be different. In other words, some figures A_i occur in the set $\mathcal{A}$ several times. In particular, $\mathcal{A}$ may consist of n copies of the same figure.

Formula (8.3) is known as the "principle of inclusions and exclusions". Since the proof of this principle is only based on axiom (γ) and the distributive law, it is true for any additive functions of a polygon, for example for the area of it (for more detail see [7]). Another example of an additive function, that of a finite set rather than of a polygon, is the number of its elements or, as we say, its power. In other words, on an n-element set this function assumes a value of n. Based on the application of the principle of inclusions and exclusions to this case is the solution of Problems 24 and 25.

Now suppose that in (8.3) all figures A_i are nonempty convex polygons. Then, according to axiom (β), each term of the sum $\sum^{(1)}$ is equal to 1. It follows from (α) and (β) that each of the terms of $\sum^{(2)}$ is either 0 or 1 depending on whether or not the intersection of a pair of convex polygons corresponding to this term is empty. Similarly, each term of $\sum^{(3)}$ is either 0 or 1 depending on whether or not the intersection of a triple of convex polygons corresponding to this term is empty, and so on. Thus we obtain the following formula for calculating the Euler characteristic for the figure A which is equal to the union

of a finite number n of nonempty convex polygons A_j:

$$\chi(A) = q_1 - q_2 + q_3 - \ldots + (-1)^{n-1} q_n. \qquad (8.10)$$

Here q_i $(i = 1, \ldots, n)$ denotes the number of i-element parts of the set $\mathscr{A} = \{A_1, \ldots, A_n\}$ such that the polygons contained in each of these parts have a nonempty intersection.

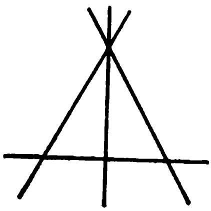


Fig. 32

Thus, if there is an Euler characteristic χ on the class $\mathscr{M}$ of all *elementary figures*, which satisfies axioms (α) to (γ), then it is *uniquely* defined: its value is given by formula (8.10). In particular, the Euler characteristic for any elementary figure is an integer.

Consider a simple example. Let a set $\mathscr{A}$ consist of four segments and let the figure A be the union of these segments (Fig. 32). In this case there are 6 pairs of line segments with a nonempty intersection, one triple of segments with a nonempty intersection, the intersection of all the four segments is empty. We have therefore $\chi(A) = 4 - 6 + 1 - 0 = -1$.

Problems

24. There are 67 people on the staff of a research institute. Of them 47 know English, 35 know German and 20 know French. In addition, it is known that 23 know both English and German, 12 know both English and French, 11 know German and French, and 5 know all the three languages. How many people at the Institute know none of the three languages?

25. How many numbers are there among the natural numbers from 1 to 1000 which are not divisible by any one of the numbers 2, 3, and 5?

26. A figure is made up of five convex polygons, all having a nonempty intersection. Find the Euler characteristic for the figure.

9. Proof of the Existence of the Euler Characteristic

We prove the existence of the Euler characteristic successively for the classes of elementary figures on a straight line, in a plane and in space.

Let $A = \bigcup_{j=1}^{m} B_j$ be an elementary figure in the class $\mathscr{M}(L)$, i.e. a union of line segments B_j (possibly, singular) lying on a straight line L. We prove that the figure A

is equal to the union of its components, i.e. such line segments C_j no two of which have any points in common. Indeed, take a line segment B_1. Two cases are possible: first, B_1 is already a component of A (and then we denote it by C_1), second, B_1 is not a component of A. In the latter case among the line segments $B_2, B_3, \ldots, B_m$, there are $B_2, \ldots, B_k$, such that each of them has at least one point in common with B_1. Then the union $C_1' = \bigcup_{j=1}^{k} B_j$ is obviously a line segment. Again two cases are possible: first C_1' is already a component of the figure A (and then we denote it $C_1' = C_1$), second, C_1' is not a component of A. In the latter case among the line segments $B_{k+1}, \ldots, B_m$, there are $B_{k+1}, \ldots, B_p$, such that each of them has at least one point in common with C_1'. Then the union $C_1'' = \bigcup_{j=1}^{p} B_j$ is a line segment. Again two cases are possible: first, C_1'' is already a component of A, second, C_1'' is not a component of A. Since there are altogether a finite number of segments B_j, this process must terminate, i.e. we single out the component C_1 of A. After that, omitting from the given system of line segments $\{B_1, \ldots, B_m\}$ those contained in C_1 and applying the process to the remaining segments we single out the components C_2, then C_3, and finally C_n. We get

$$A = \bigcup_{i=1}^{n} C_i,$$

where C_i is the component of the figure A.

We now set

$$\chi(A) = n,$$

i.e. say that the Euler characteristic for the figure A is the number of its components. To justify this name it is necessary to verify that for this function χ axioms (α) to (γ) are true. As to axioms (α) and (β) they trivially hold. It remains to prove the additivity of the function χ. Let $B = \bigcup_{j=1}^{m} D_j$ be another figure in $\mathcal{M}(L)$ with components D_j. We prove axiom (γ) in the following form:

$$\chi(A) + \chi(B) = \chi(A \cup B) + \chi(A \cap B) \qquad (9.1)$$

by induction on the number n of the components of A. We first assume that $n = 1$, i.e. that A consists of one

line segment C_1, and let C_1 have points in common with precisely k line segments, the components of the figure B, where $0 \leqslant k \leqslant m$ and m is the number of the components of B. Then the left-hand side of (9.1) is equal to $1 + m$. The first term on the right-hand side is equal to $1 + m - k$, for when we unite both figures into one, the k line segments of B are "glued" into one segment (Fig. 33).

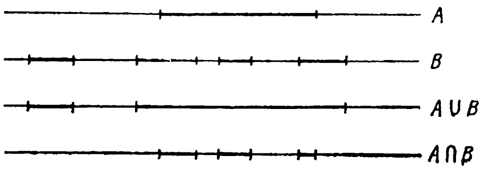


Fig. 33

The second term at the right is equal to the number k under the hypothesis. Consequently, (9.1) is true for the special case at hand. Suppose it has been proved for all figures A with at most $n - 1$ components (with the figure B fixed). We prove it for the case where A has precisely n components. We set

$$A_1 = C_1,\ A_2 = \bigcup_{i=1}^{2} C_i,\ \ldots,\ A_{n-1} = \bigcup_{i=1}^{n-1} C_i,\ A = A_n = \bigcup_{i=1}^{n} C_i.$$

Under the induction hypothesis

$$\chi(A_{n-1}) + \chi(B) = \chi(A_{n-1} \cup B) + \chi(A_{n-1} \cap B). \quad (9.2)$$

Suppose a line segment C_n intersects exactly k_n segments of the figure B and hence does not intersect the other $m - k_n$ segments of the figure. We now proceed from the figure A_{n-1} to A_n and see how both sides of (9.2) change. Clearly $\chi(A_n) - \chi(A_{n-1}) = 1$. Therefore the left-hand side of (9.2) is increased by 1. Further,

$$\chi(A_n \cup B) - \chi(A_{n-1} \cup B) = 1 - k_n,$$

for the new segment C_n "glues" into one component k_n segments of the figure B, moreover

$$\chi(A_n \cap B) - \chi(A_{n-1} \cap B) = k_n.$$

Therefore the right-hand side of (9.2) increases by $k_n + (1 - k_n) = 1$, i.e. it changes in the same way as the left-

hand side does. Hence

$$\chi(A_n)+\chi(B)=\chi(A_n \cup B)+\chi(A_n \cap B),$$

which was to be proved.

The existence of the Euler characteristic on the class $\mathcal{M}(L)$ is thus proved.

We now proceed to the case of the plane. Let $\{C_1, \ldots, C_n\}$ be a finite set of convex polygons and $A = \bigcup_{j=1}^{n} C_j$ be an elementary figure of the class $\mathcal{M}$. The polygons may contain singular polygons, i.e. segments or points. We shall assume for brevity that a point is also

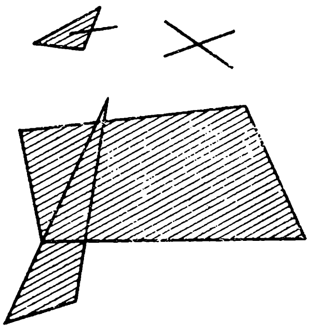


Fig. 34

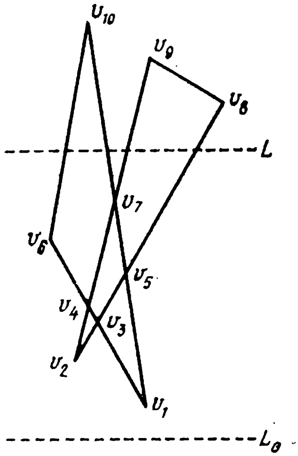


Fig. 35

a line segment (with endpoints coinciding). Consider a set T of line segments such that each of them is either a side of some polygon C_j (if the latter is nonsingular) or the polygon itself (if it is singular). By *vertices* of the figure A we shall mean, first, the endpoints of line segments that are members of the set T and, second, the points of intersection of two or more such line segments. For example, the figure in Fig. 34 has 21 vertices.

Let the vertices of a figure be all at different heights and numbered in increasing order of height, i.e. let v_1 be the lowest vertex, v_2 be above v_1 but below v_3, and so on, v_m being the uppermost vertex (Fig. 35).

We draw a horizontal straight line L_0 below the figure A and denote by h_i $(i = 1, \ldots, m)$ the distance from a vertex v_i to the straight line. According to our hypothesis $0 < h_1 < h_2 < \ldots < h_m$. Let L be a horizontal straight line moving up along the plane from its initial position L_0. The intersection $C_j \cap L$ is a line segment (possibly, singular or empty). According to the distribution law we have

$$A \cap L = \bigcup_{j=1}^{n} (C_j \cap L)$$

and, consequently, this intersection is a finite union of line segments, i.e. it is a figure of class $\mathcal{M}(L)$. Therefore there exists an Euler characteristic $\chi(A \cap L)$ for an intersection of a figure A and a moving straight line. Let $\varphi(h) = \chi(A \cap L)$ be the characteristic for a straight line L when L is in a position a distance h from its initial position L_0. Also we denote by L_i a fixed position of L when it passes through a vertex v_i $(i = 1, \ldots, m)$ and by L_{i-0} its position when it is below L_i but above L_{i-1}. It can be easily seen (Fig. 35) that when L moves and remains between two adjacent vertices (as well as below v_1 or above v_m), the number $\varphi(h)$ remains unchanged. It can only change when L "approaches" some vertex from below or "leaves" that vertex while moving up. We set

$$\varphi(h_i) = \chi(A \cap L_i), \quad \varphi(h_i - 0) = \chi(A \cap L_{i-0}) \qquad (9.3)$$

and define the Euler characteristic of the figure $A \in \mathcal{M}$ as

$$\chi(A) = \sum_{i=1}^{m} [\varphi(h_i) - \varphi(h_i - 0)]. \qquad (9.4)$$

Thus the difference $\varphi(h_i) - \varphi(h_i - 0)$ is a change in the number $\chi(A \cap L)$ when L "approaches" from below a vertex v_i (but not when it "leaves" the vertex in moving up and not when it "passes" through the vertex!) and the Euler characteristic $\chi(A)$ is the sum of such changes taken over all the vertices.

Notice that not all the differences in sum (9.4) are necessarily different from zero. For example, for the figure A (Fig. 35) we have

$$\varphi(h_1) - \varphi(h_1 - 0) = \varphi(h_2) - \varphi(h_2 - 0) = 1,$$
$$\varphi(h_3) - \varphi(h_3 - 0) = -1$$

and for all the other vertices the differences are zero. Therefore the definition gives $\chi(A) = 1$.

We prove that the function χ of A defined by (9.4) satisfies axioms (α) to (γ) and that hence it may indeed be called the Euler characteristic.

Clearly, $\chi(\varnothing) = 0$. Let A be a convex polygon. Then

$$\varphi(h_1) - \varphi(h_1 - 0) = 1 \text{ but } \varphi(h_i) - \varphi(h_i - 0) = 0$$

for every vertex v_i with index $i > 1$ (if such vertices exist). Therefore $\chi(A) = 1$. So axioms (α) and (β) hold.

We check the additivity of the function χ. Let A and B be elementary figures. We must prove the validity of the equation

$$\chi(A) + \chi(B) = \chi(A \cup B) + \chi(A \cap B). \tag{9.5}$$

Let $v_1, \ldots, v_r$ be vertices of a figure $A \cup B$ numbered in increasing order of height (notice that it suffices to consider only the set of vertices of the union $A \cup B$ since it contains vertices of the other three figures). Let the symbols L_i and L_{i-0} ($i = 1, \ldots, r$) have the same meaning as before. Each of the straight lines L_i and L_{i-0} intersects each of the figures A, B, $A \cup B$, and $A \cap B$ in a finite union of segments. For brevity we introduce the notation

$$\varphi_i(A) = \chi(A \cap L_i), \quad \varphi_{i-0}(A) = \chi(A \cap L_{i-0})$$

and similarly for the other three figures. Then, in view of additivity of the Euler characteristic on the class $\mathcal{M}(L)$ we obtain the equations

$$\varphi_i(A) + \varphi_i(B) = \varphi_i(A \cup B) + \varphi_i(A \cap B), \tag{9.6}$$

$$\varphi_{i-0}(A) + \varphi_{i-0}(B) = \varphi_{i-0}(A \cup B) + \varphi_{i-0}(A \cap B) \tag{9.7}$$

for all $i = 1, \ldots, r$. We subtract equation (9.7) term by term from equation (9.6) and sum up the equations obtained over all $i = 1, \ldots, r$. We then get

$$\sum_{i=1}^{r} [\varphi_i(A) - \varphi_{i-0}(A)] + \sum_{i=1}^{r} [\varphi_i(B) - \varphi_{i-0}(B)]$$
$$= \sum_{i=1}^{r} [\varphi_i(A \cup B] - \varphi_{i-0}(A \cup B)]$$
$$+ \sum_{i=1}^{r} [\varphi_i(A \cap B) - \varphi_{i-0}(A \cap B)],$$

and this, considering (9.3) and (9.4), means that (9.5) is true.

We have thus proved the existence of the Euler characteristic on the class $\mathcal{M}$ of elementary figures in the plane.

The proof for the class $\mathcal{M}$ (R) of elementary figures in space is similar. We use the method of a moving plane and the existence of the Euler characteristic now on the class $\mathcal{M}$.

10. The Equivalence of the Two Definitions of the Euler Characteristic

At the beginning of this section we introduce some information about binomial coefficients which is necessary in what follows.

Given a natural number n and some n-element set $A = \{a_1, \ldots, a_n\}$, any m-element part of the set is called an *m-combination* (or a *combination of m elements*) of n given elements. The number of m-combinations of n elements is denoted by $\binom{n}{m}$. For example, the 4-element set $\{a_1, a_2, a_3, a_4\}$ has six 2-element parts: $\{a_1, a_2\}$, $\{a_1, a_3\}$, $\{a_1, a_4\}$, $\{a_2, a_3\}$, $\{a_2, a_4\}$, $\{a_3, a_4\}$, and therefore $\binom{4}{2} = C_4^2 = 6$. The expression $\binom{n}{m}$ makes sense not only for $m \leqslant n$ but also for $m > n$ and is zero in this case, for an n-element set has no m-element parts at all if $m > n$. Especially notice the equation $\binom{n}{0} = 1$ which means that any n-element set has one 0-element part, namely an empty set.

Numbers $\binom{n}{m}$ are also known as *binomial coefficients* since they enter in the well-known formula

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n} x^n \quad (10.1)$$

which expresses the nth degree of the binomial $1 + x$ as a polynomial in ascending powers of x. We shall "scarcely" need (10.1).

Note the following, also well-known, formula for calculating binomial coefficients:

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!} = \frac{n(n-1)\ldots[n-(m-1)]}{1\cdot 2\ldots m}.$$

We prove the equation

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}. \quad (10.2)$$

To this end we take some n-element set $A = \{a_1, \ldots, a_n\}$ and divide all its m-element parts into two groups. We refer to the first group the parts containing the element a_n and to the second those containing no a_n. The number of parts in the first group is $\binom{n-1}{m-1}$; indeed, if we omit a_n from all these parts, we get *all* $(m-1)$-element parts of the $(n-1)$-element set $\{a_1, \ldots, a_{n-1}\}$. On the other hand, the parts of the second group are m-element parts of the same set $\{a_1, \ldots, a_{n-1}\}$; therefore their number is $\binom{n-1}{m}$. Since the two groups have no elements in common (in other words, none of the two classes contains an m-element part of the set A), equation (10.2) is proved.

An important part in what follows will be played by the formula

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\ldots+(-1)^m\binom{n}{m} = (-1)^m\binom{n-1}{m}, \tag{10.3}$$

which is true for all nonnegative integers m and n, and its special case

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\ldots+(-1)^n\binom{n}{n}=0, \tag{10.4}$$

where $m = n$. We prove (10.3) by induction on m. If $m = 0$ or $m = 1$, then (10.3) is obvious. We assume that (10.3) is true for $m = j$ and prove it for $m = j + 1$. So suppose it is known that

$$\sum_{i=0}^{j}(-1)^i\binom{n}{i}=(-1)^j\binom{n-1}{j}.$$

Then, applying (10.2), we get

$$\sum_{i=0}^{j+1}(-1)^i\binom{n}{i}=(-1)^j\binom{n-1}{j}+(-1)^{j+1}\binom{n}{j+1}$$

$$=(-1)^j\left[\binom{n-1}{j}-\binom{n}{j+1}\right]$$

$$=(-1)^j\left[-\binom{n-1}{j+1}\right]=(-1)^{j+1}\binom{n-1}{j+1}.$$

Thus (10.3) is proved. Note that (10.4) follows immediately from (10.1) if we set $x = -1$.

It is sometimes more convenient to use other forms of equations (10.3) and (10.4), namely

$$\binom{n}{1}-\binom{n}{2}+\binom{n}{3}-\ldots+(-1)^{m+1}\binom{n}{m}=1$$
$$+(-1)^{m+1}\binom{n-1}{m}, \qquad (10.5)$$

$$\binom{n}{1}-\binom{n}{2}+\binom{n}{3}-\ldots+(-1)^{n+1}\binom{n}{n}=1. \qquad (10.6)$$

We proceed to prove the equivalence of the two definitions of the Euler characteristic without calculating it. We do this for the following three classes of figures: graphs, boundaries of convex polytopes, and simple polygons in the plane that are decomposed into convex faces. It is essential in the proofs that a graph edge contains both vertices which it connects and that a face of a polytope (or of a simple polygon) contains its own boundary. It can be proved that the two definitions of the Euler characteristic (the constructive and the axiomatic one) are equivalent for any figure that can be decomposed into cells.

Given a graph G, it may be assumed that it has no isolated vertices and is therefore a union of a finite number of edges, i.e. of line segments. Generally speaking, G is in space rather than in the plane. By (8.10) its Euler characteristic is

$$\chi(G)=q_1-q_2+\ldots+(-1)^{n-1}q_n. \qquad (10.7)$$

Here q_1 denotes the total number of edges of G, q_2 is the number of pairs of its edges having a nonempty intersection or, equivalently, having a vertex in common, q_3 is the number of triples of edges with a nonempty intersection, and finally q_n is the number of n-element parts of the edge set such that all edges in each of these parts have a nonempty intersection, i.e. they have a vertex in common. We must prove the formula

$$q_1-q_2+\ldots+(-1)^{n-1}q_n=V-E. \qquad (10.8)$$

We first show the relation between the number of edges, E, and the numbers of vertices of different degrees.

Let V_1 denote the number of graph vertices of degree 1, V_2 the number of vertices of degree 2, and so on, and finally V_n the number of vertices of maximum degree n. Then

$$V_1 + 2V_2 + 3V_3 + \ldots + nV_n = 2E. \qquad (10.9)$$

(Note that in (10.8) and (10.9) n denotes in fact the same number.) Relation (10.9) is very easily obtained by summing the edges of a graph over all its vertices taking into account the fact that in this summation every edge is accounted twice.

To prove (10.8) first notice that $q_1 = E$. We now find the expression for q_2. A pair of edges with a nonempty intersection "arises", first, thanks to the presence of vertices of degree 2, one such pair corresponding to each vertex. Further, since 3 edges converge to every vertex of degree 3, there correspond to it as many pairs of edges with a nonempty intersection as there are 2-element parts in a 3-element set, i.e. $\binom{3}{2}$. Similarly, $\binom{4}{2}$ pairs of edges with a nonempty intersection correspond to every vertex of degree 4 edges and so on. Finally, $\binom{n}{2}$ such pairs correspond to every vertex of maximum degree n. Since edges intersect only at vertices, we conclude from the foregoing that

$$q_2 = \binom{2}{2} V_2 + \binom{3}{2} V_3 + \binom{4}{2} V_4 + \ldots + \binom{n}{2} V_n.$$

From similar considerations (taking into account the fact that no edge triples with a nonempty intersection "arise" from vertices of degree 2) we obtain the equation

$$q_3 = \binom{3}{3} V_3 + \binom{4}{3} V_4 + \binom{5}{3} V_5 + \ldots + \binom{n}{3} V_n.$$

Similarly,

$$q_4 = \binom{4}{4} V_4 + \binom{5}{4} V_5 + \ldots + \binom{n}{4} V_n,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$q_{n-1} = \binom{n-1}{n-1} V_{n-1} + \binom{n}{n-1} V_n,$$

$$q_n = \binom{n}{n} V_n.$$

Substituting the obtained values of q_i in (10.7) we get

$$\chi(G) = E - \left[\binom{2}{2}V_2 + \binom{3}{2}V_3 + \binom{4}{2}V_4 + \ldots + \binom{n-1}{2}V_{n-1} + \binom{n}{2}V_n\right] + \left[\binom{3}{3}V_3 + \binom{4}{3}V_4 + \ldots + \binom{n-1}{3}V_{n-1} + \binom{n}{3}V_n\right] - \left[\binom{4}{4}V_4 + \ldots + \binom{n-1}{4}V_{n-1} + \binom{n}{4}V_n\right] + \ldots + (-1)^{n-2} \times \left[\binom{n-1}{n-1}V_{n-1} + \binom{n}{n-1}V_n\right] + (-1)^{n-1}\binom{n}{n}V_n$$

or, after rearranging the terms,

$$\chi(G) = E - \binom{2}{2}V_2 + \left[-\binom{3}{2} + \binom{3}{3}\right]V_3 + \ldots + \left[-\binom{i}{2} + \binom{i}{3} - \ldots + (-1)^i\binom{i}{i-1} + (-1)^{i+1}\binom{i}{i}\right]V_i + \ldots + \left[-\binom{n}{2} + \binom{n}{3} - \ldots + (-1)^n\binom{n}{n-1} + (-1)^{n+1}\binom{n}{n}\right]V_n.$$

Applying (10.6) we get

$$\chi(G) = E - V_2 - 2V_3 - \ldots - (i-1)V_i - \ldots - (n-1)V_n.$$

Further we transform the right-hand side as follows:

$$\begin{aligned}\chi(G) &= (-E + 2E) + (V_1 - V_1) + (V_2 - 2V_2) + (V_3 - 3V_3) + \ldots + (V_n - nV_n) \\ &= -E + (V_1 + V_2 + \ldots + V_n) + (2E - V_1 - 2V_2 - 3V_3 - \ldots - nV_n).\end{aligned}$$

In view of (10.9) and the obvious equation

$$V = V_1 + V_2 + V_3 + \ldots + V_n$$

we have $\chi(G) = V - E$, which was to be proved.

Let X be a convex polytope. By the second definition the Euler characteristic of its boundary ∂X is

$$\chi(\partial X) = q_1 - q_2 + q_3 - \ldots + (-1)^{n-1} q_n, \quad (10.10)$$

where q_1 is the total number of faces of X, q_2 is the number of pairs of faces having a nonempty intersection, q_3 is the number of triples of faces having a nonempty intersection, . . ., q_n is the number of n-element parts of a set of faces such that all faces of one part have a nonempty intersection. We must prove the equation

$$q_1 - q_2 + q_3 - \ldots + (-1)^{n-1} q_n = V - E + F. \quad (10.11)$$

We first notice that always $q_1 = F$. To begin with, consider a simple case where vertices of a polytope are all of degree 3 (as in a tetrahedron, cube or dodecahedron). Then $q_2 = E$, for in this case every nonempty intersection of two faces is necessarily an edge of the polytope, and conversely all its edges have such a form. Besides, every vertex of a polytope is equal to a nonempty intersection of precisely three faces, and conversely every such intersection defines a vertex. Hence $q_3 = V$, $q_4 = q_5 = \ldots = 0$, and therefore

$$q_1 - q_2 + q_3 = F - E + V.$$

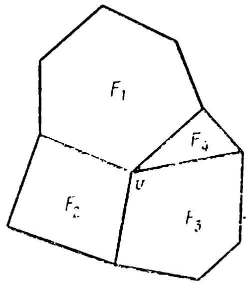


Fig. 36

This proves the special case of (10.11).

In the general case every nonempty intersection of two faces of a polytope is either its edge or its vertex. This, however, does not at all mean that $q_2 = E + V$: the same vertex may occur among such intersections several times depending on the degree of the vertex. For example, if v is a vertex of degree 4, it is contained in four different faces F_1, F_2, F_3, and F_4 of a polytope X. Fig. 36 represents the projections of the faces onto a plane rather than the faces themselves, but this does not alter the results. These faces form $\binom{4}{2} = 6$ nonempty pairwise intersections, only four of them, $F_1 \cap F_2$, $F_2 \cap F_3$, $F_3 \cap F_4$ and $F_4 \cap F_1$, being equal to edges, the other two, $F_1 \cap F_3$ and $F_2 \cap F_4$, coincide with the vertex v. In general, if v is of degree i ($i \geqslant 3$), then i faces, with the point v as their common vertex, yield $\binom{i}{2}$ nonempty pairwise intersections, $\binom{i}{1}$ intersections corresponding to edges emanating from the vertex and the remaining $\binom{i}{2}$-$\binom{i}{1}$ intersec-

tions coinciding with the vertex v. Thus we have

$$q_2 = E + \left[\binom{3}{2} - \binom{3}{1}\right] V_3 + \left[\binom{4}{2} - \binom{4}{1}\right] V_4 + \dots + \left[\binom{n}{2} - \binom{n}{1}\right] V_n, \tag{10.12}$$

where V_3 is the number of polytope vertices of degree 3, . . ., V_n is the number of polytope vertices of maximum degree n (here, just as earlier in (10.10) and (10.12), n denotes the same number). From (10.12) and the obvious equation

$$V = V_3 + V_4 + \dots + V_n = \binom{3}{0} V_3 + \binom{4}{0} V_4 + \dots + \binom{n}{0} V_n$$

we get

$$\begin{aligned} q_1 - q_2 &= F - E - \left\{\left[\binom{3}{2} - \binom{3}{1}\right] V_3 + \dots + \left[\binom{n}{2} \right.\right. \\ &\quad \left.\left. - \binom{n}{1}\right] V_n\right\} + V - \left[\binom{3}{0} V_3 + \dots + \binom{n}{0} V_n\right] \\ &= F - E + V - \left\{\left[\binom{3}{2} - \binom{3}{1} + \binom{3}{0}\right] V_3 \right. \\ &\quad \left. + \dots + \left[\binom{n}{2} - \binom{n}{1} + \binom{n}{0}\right] V_n\right\}. \end{aligned}$$

It is clear from this that to complete the proof of (10.11) it is sufficient to verify the equation

$$\left[\binom{3}{2} - \binom{3}{1} + \binom{3}{0}\right] V_3 + \dots + \left[\binom{n}{2} - \binom{n}{1} + \binom{n}{0}\right] V_n - q_3 + q_4 - \dots + (-1)^n q_n = 0. \tag{10.13}$$

Let us verify this. To do this we notice that every nonempty intersection of three, four or any larger number i of faces is necessarily a vertex of a polytope, a nonempty intersection of i faces "arising" only from vertices of degree $\geqslant i$; if a vertex is of degree m, then there are exactly $\binom{m}{i}$ such intersections equal to the vertex itself.

Therefore

$$q_3=\binom{3}{3}V_3+\binom{4}{3}V_4+\ldots+\binom{n-1}{3}V_{n-1}+\binom{n}{3}V_n,$$
$$q_4=\qquad\binom{4}{4}V_4+\ldots+\binom{n-1}{4}V_{n-1}+\binom{n}{4}V_n,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$q_{n-1}=\qquad\binom{n-1}{n-1}V_{n-1}+\binom{n}{n-1}V_n,$$
$$q_n=\qquad\binom{n}{n}V_n.$$

Substituting these expressions in (10.13) and changing the order of terms we see that the left-hand side of (10.13) is

$$\left[\binom{3}{0}-\binom{3}{1}+\binom{3}{2}-\binom{3}{3}\right]V_3+\left[\binom{4}{0}-\binom{4}{1}+\binom{4}{2}-\binom{4}{3}\right.$$
$$\left.+\binom{4}{4}\right]V_4+\ldots+\left[\binom{n-1}{0}\right.$$
$$-\binom{n-1}{1}+\ldots+(-1)^{n-1}$$
$$\left.\times\binom{n-1}{n-1}\right]V_{n-1}+\left[\binom{n}{0}\right.$$
$$-\binom{n}{1}+\ldots+(-1)^{n-1}$$
$$\left.\times\binom{n}{n-1}+(-1)^n\binom{n}{n}\right]V_n.$$

In view of (10.4) every square bracket is zero, proving (10.13) and hence (10.11).

Let M be a simple polygon decomposed into convex faces. We prove the equation

$$q_1-q_2+q_3-\ldots+(-1)^{n-1}q_n=V-E+F, \qquad (10.14)$$

where we use the same notation as in (10.11) but it refers to an arbitrary partition M. The proof follows the same procedure. Notice, however, some differences.

By the *degree* of a vertex of a partition we mean the number of edges emanating from it. In contrast to the case of the polytope, now the vertices (as well as edges) are divided into interior and boundary ones. Let V^i be the total number of interior vertices, V_3^i be the number of interior vertices of degree 3, . . ., V_n^i be the number of interior vertices of maximum degree n. Then

$$V^i=V_3^i+V_4^i+\ldots+V_n^i=\binom{3}{0}V_3^i+\binom{4}{0}V_4^i+\ldots+\binom{n}{0}V_n^i. \qquad (10.15)$$

Further, let V^e be the number of all boundary vertices, and E^i and E^e be the numbers of interior and boundary edges respectively. Obviously

$$V = V^i + V^e,\ E = E^i + E^e,\ V^e = E^e. \tag{10.16}$$

Also we denote by $V_3^e, \ldots, V_n^e$ the number of boundary vertices of degree 3, . . ., n respectively.

As in the case of a polytope we have $q_1 = F$. Even in calculating q_2, however, distinctions appear: first, only interior edges are intersections of faces while boundary edges are not; second, now $j-1$ rather than j faces converge to a boundary vertex of degree j ($j \geqslant 3$). Hence such a vertex yields $\binom{j-1}{2}$ nonempty pairwise intersections of faces, $j-2$ intersections corresponding to the interior edges emanating from this vertex and the remaining $\binom{j-1}{2}$-$(j-2)$ intersections being equal to the vertex itself. On the other hand, just as before, every interior vertex v of degree j gives $\binom{j}{1}$ nonempty pairwise intersections of faces, each intersection corresponding to an interior edge, and $\binom{j}{2}$-$\binom{j}{1}$ nonempty pairwise intersections coinciding with the vertex v itself. Therefore

$$\begin{aligned} q_1 - q_2 = F - E^i & \\ - \Big\{\Big[\binom{3}{2} - \binom{3}{1}\Big] V_3^i & + \Big[\binom{4}{2} - \binom{4}{1}\Big] V_4^i \\ + \ldots + \Big[\binom{n}{2} - \binom{n}{1}\Big] V_n^i & + \Big[\binom{2}{2} - 1\Big] V_3^e + \Big[\binom{3}{2} - 2\Big] V_4^e \\ + \ldots + \Big[\binom{n-1}{} - (n-2)\Big] V_n^e \Big\}. & \end{aligned} \tag{10.17}$$

From (10.16) we get

$$F - E^i = (F - E + V) + E^e - V^i - V^e = (F - E + V) - V^i. \tag{10.18}$$

In view of (10.17) and (10.18), to complete the proof of (10.14) it is sufficient to verify that

$$\begin{aligned} V^i + \Big[\binom{3}{2} - \binom{3}{1}\Big] V_3^i + \ldots + \Big[\binom{n}{2} - \binom{n}{1}\Big] V_n^i & \\ + \Big[\binom{2}{2} - 1\Big] V_3^e + \ldots + \Big[\binom{n-1}{2} - (n-2)\Big] V_n^e & \\ - q_3 + q_4 - \ldots + (-1)^n q_n = 0. & \end{aligned} \tag{10.19}$$

To verify this we find the numbers q_3, q_4, and so on. Since nonempty triple face intersections "arise" from interior vertices of degree $\geqslant 3$ and from boundary vertices of degree $\geqslant 4$, we have

$$\begin{aligned} q_3 = \binom{3}{3} V_3^i + \binom{4}{3} V_4^i + \ldots + \binom{n-1}{3} V_{n-1}^i + \binom{n}{3} V_n^i & \\ + \binom{3}{3} V_4^e + \ldots + \binom{n-2}{3} V_{n-1}^e + \binom{n-1}{3} V_n^e. & \end{aligned}$$

Similarly,

$$q_4 = \binom{4}{4} V_4^i + \binom{5}{4} V_5^i + \ldots + \binom{n-1}{4} V_{n-1}^i + \binom{n}{4} V_n^i$$
$$+ \binom{4}{4} V_5^e + \ldots + \binom{n-2}{4} V_{n-1}^e + \binom{n-1}{4} V_n^e,$$

. . . .

$$q_{n-1} = \binom{n-1}{n-1} V_{n-1}^i + \binom{n}{n-1} V_n^i + \binom{n-1}{n-1} V_n^e,$$

$$q_n = \binom{n}{n} V_n^i.$$

Taking into account (10.15) and substituting the numbers $q_3, q_4, \ldots, q_n$ in the left-hand side of (10.19) we find that after simple transformations it has the form

$$\left[\binom{3}{0} - \binom{3}{1} + \binom{3}{2} - \binom{3}{3}\right] V_3^i + \left[\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4}\right] V_4^i$$
$$+ \ldots + \left[\binom{n}{0} - \binom{n}{1} + \ldots + (-1)^n \binom{n}{n}\right] V_n^i$$
$$+ \left[-1 + \binom{2}{2}\right] V_3^e + \left[-2 + \binom{3}{2} - \binom{3}{3}\right] V_4^e$$
$$+ \ldots + \left[-(n-2) + \binom{n-1}{2} - \binom{n-1}{3}\right.$$
$$\left. + \ldots + (-1)^{n-1} \binom{n-1}{n-1}\right] V_n^e.$$

All coefficients of the terms $V_3^i, V_4^i, \ldots, V_n^i$ are zero in view of (10.14). As to the coefficients of $V_3^e, V_4^e, \ldots, V_n^e$, each of them is also zero. The easiest way to show this is to use (10.6) rewriting it as follows:

$$\binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \ldots + (-1)^n \binom{n}{n} = \binom{n}{1} - 1 = n - 1$$

This proves (10.19) and hence (10.14).

Problem

27. A figure is a union of n convex polygons $A_1, \ldots, A_n$, with $\bigcap_{i=1}^{n} A_i \neq \varnothing$. Find the Euler characteristic for the figure.

11. Elementary Figures on the Sphere and Their Euler Characteristics

Let S be a sphere. A *great circle* on S is the line of intersection of the sphere and a plane passing through its centre O (Fig. 37*a*). It divides the sphere into two *hemispheres.* We shall assume that a great circle itself belongs

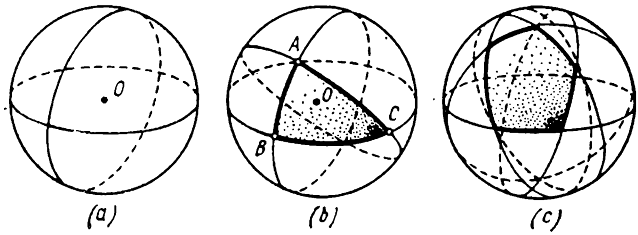


Fig. 37

to each of them. A *convex polygon* on S is the intersection of a finite number of hemispheres (Fig. 37*b*, *c*). Thus convex polygons are defined on a sphere the same way as in the plane, but the role of straight lines here is played by great circles and that of half-planes by hemispheres.

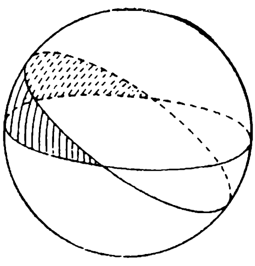


Fig. 38

In contrast to the plane, where a triangle is a polygon with the smallest number of sides, there are convex polygons on a sphere with the number of sides less than three, the lunes. A *lune* is the intersection of two hemispheres whose boundary great circles do not coincide (Fig. 38). The great circle is also a convex polygon since it is the intersection of two hemispheres (it defines). Finally, among convex polygons on S are pairs of *antipodal points*, i.e. points on a sphere at opposite ends of a diameter. Indeed, a pair of antipodes is the intersection of two great circles and hence the intersection of four hemispheres.

A convex polygon on a sphere is said to be *strictly convex* if it contains not a single pair of antipodes. Such, for example, are the triangle and the pentagon in Fig. 37.

On the contrary, a lune is not strictly convex since it contains a (single) pair of antipodes, its vertices. A convex polygon is said to be *singular* if it is in the great circle (in particular, if it coincides with the circle).

An *elementary figure* on a sphere is a union of a finite number of *strictly convex* polygons, possibly singular. This class of figures is denoted by $\mathscr{M}(S)$. It can be easily seen that $\mathscr{M}(S)$ contains not only strictly convex polygons but *all* convex polygons (for example, a hemisphere, a lune, and a pair of antipodes). It is of importance to note that the sphere S is among this class. Indeed, let v_1, v_2, v_3 and v_4 be the vertices of a tetrahedron inscribed in S such that the centre of the sphere is inside it (Fig. 39). Consider four triangles on the sphere: A_1 with vertices v_2, v_3, and v_4, A_2 with vertices v_1, v_3, and v_4, A_3 with vertices v_1, v_2, and v_4, and A_4 with vertices v_1, v_2, and v_3. All these triangles are clearly strictly convex, and in addition

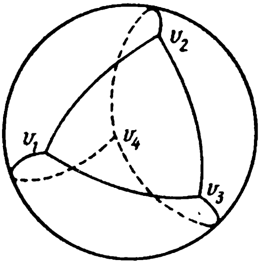


Fig. 39

$$S=\bigcup_{i=1}^{4} A_i. \tag{11.1}$$

Let C be the great circle of a sphere. An *elementary figure* on C is the union of a finite number of arcs whose lengths are smaller than the length of a semicircle. The shorter arcs are called *minor arcs*. In particular, the circle itself can be represented as the union of three minor arcs each two of which have an endpoint in common:

$$C=\bigcup_{i=1}^{3} B_i. \tag{11.2}$$

Let $\mathscr{M}(C)$ denote the class of all elementary figures on the circle.

The Euler characteristic for $\mathscr{M}(S)$ and $\mathscr{M}(C)$ is defined using axioms (α), (β), and (γ) of Sec. 8. In contrast to the case of the plane, however, axiom (β) is now stated as follows:

(β) for every (including singular) nonempty *strictly convex* polygon A on S we have $\chi(A) = 1$.

In particular, for every nonempty minor arc B of C we have $\chi(B) = 1$.

The reason why $\chi(A) = 1$ must now hold only for strictly convex (rather than all convex) polygons is clear: as a matter of fact, among convex polygons is, for example, a pair of antipodes, and it is natural to consider the number 2, not 1, its Euler characteristic.

The proof of the existence of the Euler characteristic for the class $\mathcal{M}(C)$ is similar to that already carried out for the class of elementary figures on a straight line, but with one difference. Namely, it is easy to verify that any elementary figure M on the circle C is either equal to the union of a finite number of arcs (not necessarily minor) of the circle which pairwise have no points in common and are usually called the components of the figure M or coincides with the circle C. In the first case $\chi(M)$ is assumed to be equal to the number of the components of the figure M. In the second case we must assume that $\chi(M) = \chi(C) = 0$ since this equation follows necessarily from the axioms of the Euler characteristic, its uniqueness, and representation (11.2). Indeed,

$$\chi(C) = \sum^{(1)} \chi(B_i) - \sum^{(2)} \chi(B_{i_1} \cap B_{i_2})$$
$$+ \chi(B_1 \cap B_2 \cap B_3) = 3 - 3 + 0 = 0.$$

Just as in Sec. 9 it can be verified that axioms (α) to (γ) are satisfied.

The proof of the existence of the Euler characteristic for the class $\mathcal{M}(S)$ is similar to that carried out for the class of elementary figures in the plane, i.e. we use the method of a "rotating" great circle; however there are some differences here too. Let

$$M = \bigcup_{j=1}^{n} A_j$$

be an elementary figure of the class $\mathcal{M}(S)$, i.e. a union of strictly convex polygons A_j on a sphere. These may contain singular polygons, i.e. minor arcs or points. Two cases are possible: $M = S$ and $M \neq S$. In the first case we must assume that $\chi(M) = \chi(S) = 2$, which necessarily follows from the axioms of the Euler characteristic,

its uniqueness and representation (11.1):

$$\chi(S)=\sum^{(1)}\chi(A_j)-\sum^{(2)}\chi(A_{j_1}\cap A_{j_2})$$
$$+\sum^{(3)}\chi(A_{j_1}\cap A_{j_2}\cap A_{j_3})$$
$$-\chi(A_1\cap A_2\cap A_3\cap A_4)=4-6+4-0=2.$$

In the second case we take a pair of antipodal points N_1 and N_2 distinct from all vertices of the figure M. We call the points N_1 and N_2 the "poles" (say, the "north" and the "south" pole). We choose a great circle C_0 so that it should pass through the poles but should not pass through the vertices of M (of which there are a finite number). Let, for example, C_0 consist of the "Greenwich meridian" and a "date-change line". Let C be the great circle passing through the poles and "rotating", say, from "west" to "east" from the initial position C_0 to the final position also C_0 (but in such a way that the "Greenwich meridian" and the "date-change line" interchange places).

We assume

$$\chi(M)=\chi(M\cap N_1)+\chi(M\cap N_2)+\sum_{i=1}^{m}[\chi(M\cap C_i)$$
$$-\chi(M\cap C_{i-0})]. \tag{11.3}$$

Here C_i denotes the position of the rotating circle at the moment when it passes through some vertex v_i of the figure M and C_{i-0} is just to the "west" of C_i. Just as in Sec. 9, we can prove that the function χ of M defined by (11.3) satisfies the axioms of the Euler characteristic, without the requirement that in each position the circle C should meet only one vertex of M.

Problems

28. A football is usually made from pieces of leather of two shapes: pentagonal and hexagonal (differing in colour as well as in shape). Is it possible to make a football using only hexagonal pieces?

29. There are n ($n \geqslant 3$) great circles on a sphere which do not pass through the same pair of antipodal points. Is there a point on the sphere that lies exactly on two of these circles?

12. Further Application of the Euler Characteristic

In this section we use the formula

$$\chi(M) = c(M) - c^*(M) + 1 \tag{12.1}$$

for polygons M in the plane or on a sphere. Recall that $c(M)$ denotes the number of the components of a figure M and $c^*(M)$ the number of the components of its complement with respect to the plane or sphere.

Notice first without proof that any polygon B can be uniquely represented as a union

$$B = \bigcup_{i=1}^{m} A_i \tag{12.2}$$

of polygons A_i such that each of them is either simple or has only simple holes, the intersection of each pair $A_i \cap A_j$ either consisting of a single point or being empty.

For example, polygon (c) (Fig. 8 on page 19) is made up of three triangles, polygon (f) is made up of two triangles and one polygon with two simple holes, and polygon (g) is made up of three simple polygons and one polygon with three simple holes.

The uniqueness of representation (12.2) is only ensured in general by the above restrictions on the intersection of pairs $A_i \cap A_j$. Indeed, polygon (e) of Fig. 8 may be regarded as a union of three simple polygons each of which has two points in common, however. On the other hand, this polygon (e) may be regarded as a unique "component" of itself in representation (12.2) but having three simple holes.

We proceed to prove (12.1). Let at first a polygon B have no holes. We prove that then $\chi(B) = 1$. Indeed, in (12.2) the "components" of A_j can be arranged so that all the figures

$$B_1 = A_1,\quad B_2 = A_1 \cup A_2,\quad B_3 = \bigcup_{i=1}^{3} A_i,\ \ldots,\ B = B_m = \bigcup_{i=1}^{m} A_i$$

should be connected (Fig. 40). Then for all subscripts i the intersection $B_{i-1} \cap A_i$ consists of a single point, for otherwise the polygon B_i would have holes (Fig. 41).

Therefore

$$\chi(B_1) = \chi(A_1) = 1,$$
$$\chi(B_2) = \chi(A_1) + \chi(A_2) - \chi(A_1 \cap A_2) = 1 + 1 - 1 = 1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\chi(B) = \chi(B_{m-1}) + \chi(A_m) - \chi(B_{m-1} \cap A_m) = 1 + 1 - 1 = 1.$$

On the other hand, for a polygon without holes we have $c(B) = c^*(B) = 1$, and hence (12.1) is true.

Now let a polygon M have n holes $C_1, \ldots, C_n$ and let them all be simple. We prove (12.1) by induction on the number of holes.

Let M_0 be a polygon obtained from M by "gluing up" all the holes. We know that (12.1) is true for M_0. Let M_1

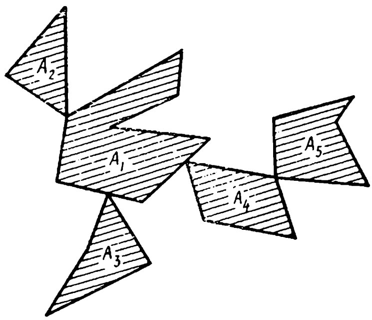


Fig. 40

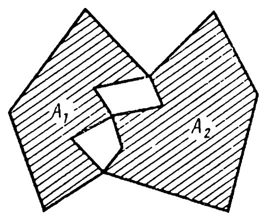


Fig. 41

be a polygon obtained from M_0 by "cutting out" a hole C_1, M_2 a polygon obtained by cutting out a hole C_2, and so on, $M_n = M$ be a polygon obtained from M_{n-1} by cutting out a hole C_n. Clearly

$$c(M_1) = c(M_2) = \ldots = c(M_n) = 1,$$
$$c^*(M_1) = 2, \quad c^*(M_2) = 3, \ldots, c^*(M_n) = c^*(M) = n + 1.$$

Suppose (12.1) is true for the polygon M_{n-1}, i.e. let

$$\chi(M_{n-1}) = c(M_{n-1}) - c^*(M_{n-1}) + 1 = 1 - n + 1 = 2 - n.$$

We prove a similar equation for $M_n = M$. We have $M_{n-1} = M_n \cup C_n$ from which $\chi(M_{n-1}) = \chi(M_n) + \chi(C_n)$. Moreover, by the corollary to Theorem 1 $\chi(C_n) = 1$.

Therefore

$$\chi(M_n) = \chi(M_{n-1}) - \chi(C_n) = 2 - n - 1 = 1 - n.$$

On the other hand,

$$c(M_n) - c^*(M_n) + 1 = 1 - n - 1 + 1 = 1 - n,$$

proving equation (12.1) for the polygon having only simple holes.

In the most general case this equation is obtained from the additive property of the Euler characteristic and representation (12.2). Notice that the proof for the sphere is the same, but it should be taken into account that in contrast to the plane case gluing up holes may result in a figure coinciding with the entire sphere.

The Euler characteristic is most conveniently applied when the properties of a covering of a figure (for example, a sphere) by a family of convex polygons are related to those of the intersection of the covering family. A family $\{A_1, \ldots, A_n\}$ of convex polygons on a sphere S is said to *cover* the sphere if

$$S = \bigcup_{i=1}^{n} A_i.$$

Theorem 3. *Let A_1, A_2, A_3 and A_4 be nonsingular strictly convex polygons on a sphere S. Then the following statements are equivalent:*

(a) *the polygons cover the sphere,*

(b) *the intersection of all the four polygons is empty and the intersection of every three of them is nonempty.*

We formulate a similar proposition for the great circle (instead of the sphere).

Theorem 4. *Let A_1, A_2, and A_3 be minor arcs on a great circle C. Then the following statements are equivalent:*

(a) *the arcs cover the circle,*

(b) *the intersection of all three arcs is empty and the intersection of every two arcs of them is nonempty.*

We prove Theorem 3 using Theorem 4 as a *lemma*. Since Theorems 3 and 4 are proved in the same way, the proof of Theorem 4 will be left to the reader.

Proof. We prove that (a) implies (b). Let four polygons cover a sphere. It is then clear that there cannot be a point on the sphere that would be shared by all the polygons. If there were such a point, its antipode would not belong to any one of the polygons because they are strict-

ly convex, and this contradicts the hypothesis. We prove that every three polygons of the given four have a point in common. To do this we take the polygon A_4 and draw on the sphere a great circle C which has no points in common with A_4. That such a circle exists can be shown as follows. The polygon A_4 is equal to the intersection of a certain number of hemispheres. If we take only two among these hemispheres, then their intersection is a lune D. We draw a great circle C' which has only two points in common with D, its vertices v and v'. To obtain A_4 from D, it is necessary to "cut" some pieces from D. In particular, at least one of the vertices v and v' will be cut off. But then C' might be rotated so that it has no points in common with A_4.

Thus there is such a great circle C that $C \cap A_4 \neq \varnothing$. In such a case the circle C is covered by three sets $A_1 \cap C$, $A_2 \cap C$, and $A_3 \cap C$ that are minor arcs. By Theorem 4 every two of the three arcs have a point in common. Hence every pair of three polygons A_1, A_2, and A_3 have also a point in common. Since instead of A_4 we could take any other polygon, any two of the four polygons have a nonempty intersection. Therefore $\chi(A_i \cap A_j) = 1$ for all $i, j = 1, 2, 3, 4$. Now, using the equation $S = \bigcup_{i=1}^{4} A_i$ and the formula (8.10) we get

$$2 = \chi(S) = q_1 - q_2 + q_3 - q_4 = \binom{4}{1} - \binom{4}{2} + q_3 - 0$$

from which $q_3 = 4$. This means that *every* three polygons taken from the set $\{A_1, A_2, A_3, A_4\}$ have a nonempty intersection since there are exactly 4 such triples. Thus we have proved that (a) implies (b).

We now prove that (b) implies (a). Suppose now statement (b) holds. Set $A = \bigcup_{i=1}^{4} A_i$. Then

$$\chi(A) = q_1 - q_2 + q_3 - q_4 = \binom{4}{1} - \binom{4}{2} + \binom{4}{3} - 0 = 2.$$

The figure A is connected, hence $c(A) = 1$. Applying to the figure formula (12.1), which is also true for the sphere, we get $c^*(A) = 0$ which means that A covers the sphere S. Theorem 3 is proved.

Corollary. *The smallest number of strictly convex polygons covering a sphere is 4.*

Proof. Indeed, (11.1) shows that there are four strictly convex triangles covering a sphere. On the other hand, if it is covered by three strictly convex polygons A_1, A_2, and A_3, then it is also covered by four polygons A_1, A_2, A_3, A_4. Then by Theorem 3 the intersection $\bigcap\limits_{i=1}^{3} A_i$ is nonempty, which contradicts the theorem.

Theorem 5. *Let A_1, A_2, and A_3 be lunes on a sphere S. Then the following statements are equivalent:*

(a) *the lunes cover the sphere,*

(b) *the intersection of all the three lunes is equal to a pair of antipodes and the intersection of every two of them is nonempty and distinct from a pair of antipodes.*

Proof. First notice that the Euler characteristic for a lune as well as for any nonempty convex polygon contained in it and different from a pair of antipodes is equal to 1 (verify it!).

We prove that (a) implies (b). Let the lune A_1, A_2, and A_3 cover the sphere S and let $A_1 \cap A_2 \neq \varnothing$. We

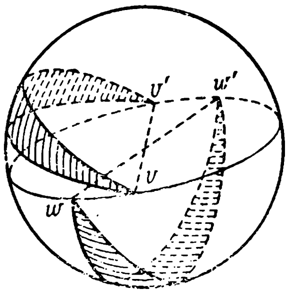


Fig. 42

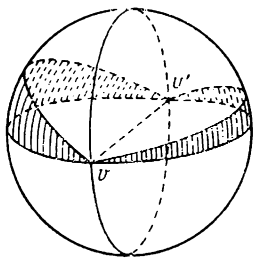


Fig. 43

denote by v and v' the vertices of A_1 and by w and w' the vertices of A_2. Under the hypothesis all the four points v, v', w, and w' are distinct. It is possible therefore to draw a unique great circle C on S through them (Fig. 42). Then the intersection $A_1 \cap C$ consists of only two points, v and v'. Indeed, otherwise $A_1 \cap C$ would contain one half of C and one of the vertices w or w' which A_1 would contain lies on that half, and this is impossible. Thus we have shown that $A_1 \cap A_2 = \varnothing$. It can be verified in the same way that the intersection of every pair of lunes is nonempty. We show in addition that the inter-

section is distinct from a pair of antipodes. Let on the contrary $A_1 \cap A_2 = \{v_1, v'\} = \{w, w'\}$. In such a case it is possible to draw on S a great circle which intersects each of the lunes A_1 or A_2 only in its vertices v and v' (Fig. 43). But then all the points of the circle, except v or v', must be contained in A_3, which is impossible.

Thus, if the lunes A_1, A_2, and A_3 cover the sphere, then the intersection of every two of them is nonempty and differs from a pair of antipodes and hence has an Euler characteristic equal to 1. Now we have

$$2 = \chi(S) = \chi\left(\bigcup_{i=1}^{3} A_i\right) = \sum{}^{(1)}\chi(A_i) - \sum{}^{(2)}\chi(A_{i_1} \cap A_{i_2})$$
$$+ \chi(A_1 \cap A_2 \cap A_3) = 3 - 3 + \chi(A_1 \cap A_2 \cap A_3).$$

Hence $\chi(A_1 \cap A_2 \cap A_3) = 2$ and therefore this intersection is a pair of antipodes. This proves that (a) implies (b).

We now prove that (b) implies (a).

Suppose statement (b) is satisfied. We set $A = \bigcup_{i=1}^{3} A_i$, then

$$\chi(A) = \sum{}^{(1)}\chi(A_i) - \sum{}^{(2)}\chi(A_{i_1} \cap A_{i_2})$$
$$+ \chi(A_1 \cap A_2 \cap A_3) = 3 - 3 + 2 = 2.$$

The figure A is connected, hence $c(A) = 1$. From (12.1) we then get $c^*(A) = 0$. Consequently, A covers the sphere S. Theorem 5 is proved.

Corollary. *The smallest number of lunes covering a sphere is 3.*

Problems

30. Give an example of an elementary figure in space for which (12.1) is false.

31. Let A_1, A_2, and A_3 be convex polygons in the plane every two of which have a nonempty intersection. Prove that if their union is convex, then the intersection of all the three of them is nonempty.

32. Let A_1, A_2, A_3, and A_4 be convex polygons in the plane such that every three of them have a point in common. Using (12.1) prove that all the polygons have a point in common.

The statement of this problem is a special case of the theorem proved in 1913 by the Austrian mathematician E. Helly (1884-1943), see [2].

33. Given four convex polygons in the plane, the intersection of each two of them being nonempty and the union of every three

having a connected complement with respect to the plane, prove that all the polygons have a point in common.

34. Prove formula (12.1) for a plane figure equal to the union of a finite number of segments.

35. Let $A_1, \ldots, A_n$ be segments in the plane such that the intersection of every two of them is nonempty and the intersection of every three is empty. Into how many parts is the plane divided by them?

36. Let A be a plane figure equal to the union of n segments. Prove that

$$\frac{n(3-n)}{2} \leqslant \chi(A) \leqslant n.$$

37. Let there be five strictly convex polygons on a sphere, each three having a nonempty intersection. Prove that some four of them also have a nonempty intersection.

38. Prove that among any five (or more) strictly convex polygons on a sphere there are four such that do not cover the sphere.

39. Let a segment A be covered by "small" segments $A_1, \ldots, A_n$, the covering being *pointwise-odd*, i.e. such that every point of A belongs to an odd number of small segments. Prove that n is odd.

Using this prove that if a figure B lying on a straight line is pointwise-odd covered by some n segments, then the difference $n - \chi(B)$ is even.

40. Let a circle be pointwise-odd covered by a finite number of short arcs. Prove that the total number of arcs is even.

Solutions, Hints and Answers

1. Add $2V$ to both sides of $E - V = 1$.

2. Add $2E$ to both sides of $V - E + F = 1$.

3. Draw a straight line with index i $(1 \leqslant i \leqslant n)$ through the points of the plane with Cartesian coordinates $(i, 0)$ and $(0, n - 1)$. Find the coordinates of the point of intersection of each pair of straight lines. Deduce from this that these points are different for different pairs of straight lines.

4. Since every two straight lines have a point in common, $V = \frac{n(n-1)}{2}$. It is clear that each vertex is of multiplicity 2. Therefore from (1.3) and (1.4) we get

$$E = n + \frac{n(n-1)}{2} \cdot 2 = n^2,$$

$$F = 1 + n - \frac{n(n-1)}{2} + \frac{n(n-1)}{2} \cdot 2 = 1 + n + \frac{n(n-1)}{2}.$$

5. Point out what changes would appear in the proof of (1.2) if, for example, the straight line L_1 of the family is horizontal. Let m vertices $A_1, \ldots, A_m$ with multiplicities $\alpha_1, \ldots, \alpha_m$ lie on it. We find the number of new vertices, edges, and faces which "arise" when the moving straight line passes through L_1. Clearly the number of new vertices is m, the points $A_1, \ldots, A_m$. The number of new edges is $(m + 1) + (\alpha_1 - 1) + \ldots + (\alpha_m - 1)$, $m + 1$ edges lying on L_1, $\alpha_1 - 1$ edges having A_1 as their lower endpoint, $\alpha_2 - 1$ edges having A_2 as their lower endpoint, and so on. The number of new faces is $(m + 1) + (\alpha_1 - 2) + \ldots + (\alpha_m - 2)$, $m + 1$ faces adjoining L_1 with their edges, $\alpha_1 - 2$ faces having A_1 as their lowest vertex, $\alpha_2 - 2$ faces having A_2 as their lowest vertex, and so on. Therefore the change in the alternating sum $V - E + F$ when the moving straight line passes through L_1 is

$$m - [(m+1) + (\alpha_1 - 1) + \ldots + (\alpha_m - 1)] + [(m+1) + (\alpha_1 - 2) + \ldots + (\alpha_m - 2)] = 0.$$

Otherwise the proof remains the same.

6. Let there be a family of n straight lines. If $n = 1$, then $V = 0$, $E = 1$, $F = 2$, and therefore $V - E + F = 1$. Suppose this formula has been proved for all families consisting of $n - 1$ straight lines in general position. Add to such a family a new straight line L. Since it intersects all the other straight lines, the number of new vertices is $n - 1$. The number of new edges is $n + (n - 1)$, n of them lying on L and the other $n - 1$ lines lying one on each of the other straight lines. Since L is divided by the other straight lines into n parts, it intersects n old faces and divides each of them into two parts. Hence the number of new faces is n. When the straight line L is added, the change in the sum $V - E + F$ is therefore $(n - 1) - [n + (n - 1)] + n = 0$.

7. If the partition of the plane is formed by a family of n straight lines and has vertices, then each straight line of the family intersects some other straight line. Therefore there are two rays, i.e. unbounded edges on each straight line. In other words, $E_2 = 2n$. Describe in the plane a circle of so large a radius that all bounded faces and edges should lie inside it. Then, when we move along that circle we shall meet in turn unbounded faces and unbounded edges. This means that $F_2 = 2n$. From (1.3) and (1.4) we get

$$E_1 = E - E_2 = -n + \sum_{i=1}^{V} \alpha_i, \quad F_1 = F - F_2 = 1 - n - V + \sum_{i=1}^{V} \alpha_i.$$

From this we at once get $V - E_1 + F_1 = 1$.

8. A partition of the plane by a family of straight lines has bounded faces if and only if the family contains three straight lines in general position or four straight lines that are the extensions of the sides of a parallelogram. There are bounded edges if and only if the family contains three straight lines in general position or three straight lines that are the extensions of three sides of a parallelogram.

9. Let x and y be two points in M, x lying in a bounded face M_1 and y in a bounded face M_2. Both faces are convex polygons. Since M_1 and M_2 are bounded, there are *intersecting* straight lines L_1 and L_2 such that L_1 is the extension of some side of M_1 and L_2 is the extension of some side of M_2. If x lies in the *interior* of M_1, then drop from it a perpendicular to any side of M_1, and do the same for y. It is now easy to draw a broken line connecting x and y. It consists of segments of the perpendiculars, some parts of the boundaries of M_1 and M_2 and of the two segments lying on L_1 and L_2 respectively. So the figure M is connected and is therefore a polygon.

The boundary of M is a closed broken line N. Number its vertices $A_1, \ldots, A_n$ by choosing an arbitrary direction for tracing the boundary. Let N have a point of self-intersection, i.e. let, for example, $A_1 = A_m = A_n$, where $1 < m < n$. It may be assumed that $3 < m < n-2$ and that m is the smallest of the numbers k $(3 < k < m - 2)$ for which $A_k = A$. Then $A_1A_2 \ldots A_{m-1}A_m$ is the circuit bounding the simple polygon M_1 contained in M. Suppose in addition that the three points A_{m-1}, $A_1 = A_m$, and A_{m+1} lie on the same straight line in the indicated order, and so do the points A_{n-1}, $A_n = A_1$, and A_2.

We prove by reductio ad absurdum that $m = 4 = n - 3$. Let, for example, $m > 4$. Then there are two straight lines in a family of straight lines that do not pass through A_1, a straight line L_1 passing through A_2 and a straight line L_2 passing through A_{m-1}. Consider another straight line of the family, L_3, passing through A_{m+1}, rather than through A_1. It intersects at least one of the straight lines L_1 and L_2, say L_1 at the point A. The broken line $A_1A_2AA_{m+1}A_1$ bounds some polygon M_2 in M. Consider the angles $A_2A_1A_{m-1}$ and $A_2A_1A_{m+1}$. They are interior for the polygons M_1 and M_2 respectively and adjacent to each other. Therefore the segment A_1A_2 intersects the interior of the polygon M, which is impossible. So $m = 4$, and similarly $n - 3 = 4$, i.e. $n = 7$, the straight lines A_2A_3 and A_5A_6 being parallel, for otherwise, denoting them by L_1 and L_3, we may repeat the reasoning.

Thus if the boundary of M has a point of self-intersection, then M itself is of the form represented in Fig. 44, where $A_1 = A_4 = A_7$. Notice that the triangles $A_1A_2A_3$ and $A_1A_5A_6$ are not necessarily faces of the partition since there may be straight lines in the family that are parallel to A_2A_3 and A_5A_6 and lie between them.

11. We prove by reductio ad absurdum. Let all the angles of all the faces be $<2\pi/5$. Then all the faces are triangles, because for $n \geqslant 4$ the largest angle of an n-gon is at least $\frac{(n-2)\pi}{n} \geqslant \frac{\pi}{2}$. Let F be the total number of faces and E_1 the number of interior edges. Since all the faces are triangles we have

$$F \leqslant \frac{2}{3} E_1. \qquad \text{(S.1)}$$

It follows from the assumption that each interior vertex is adjoined by $\geqslant 6$ edges. Therefore if V_1 is the number of interior vertices, then

$$V_1 \leqslant \frac{2}{6} E_1 = \frac{1}{3} E_1. \qquad \text{(S.2)}$$

From Euler's formula $(V_1 + 5) - (E_1 + 5) + F = 1$ and from (S.1) and (S.2) we get

$$\frac{1}{3} E_1 - E_1 + \frac{2}{3} E_1 \geqslant 1,$$

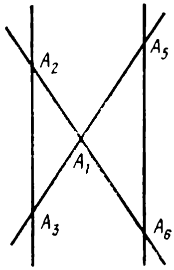


Fig. 44

which is impossible.

13. If a complete graph with five vertices is embedded in the plane, then a simple polygon M is obtained on it partitioned into faces that are also simple polygons. By the hypothesis there are 5 vertices and $\frac{5(5-1)}{2} = 10$ edges in the partition. We apply Euler's formula in the following form: $V - E + F = 2$ where the number of faces includes the complement of M with respect to the plane or an unbounded "face". Hence $F = 7$. Since two vertices of the graph may be connected by only one of its edges, there are at least three edges on the boundary of each (including unbounded) face. Therefore $3F \leqslant 2E$; hence $F \leqslant \frac{2}{3} E = \frac{20}{3} < 7$.

14. Inequality (5.10) is obtained from Pick's formula (5.1) with regard for the inequality $L \geqslant b$. If the polygon is divided into squares, then $L = b$.

15. The first formula is proved the way (5.7) is, taking into account the fact that the area of a primitive triangle is $\frac{1}{8}$. The second formula follows from the first and from (5.7) by eliminating the terms $-\chi(M) + \frac{1}{2}\chi(\partial M)$.

17. Choose vertices $v_1, \ldots, v_n$ of a polytope X lying at different heights and numbered in increasing order so that the height of each remaining vertex coincides with height of the one of the vertices chosen. Consider the sum $S(h) = V(h) - E(h) + F(h)$, where $V(h)$ is the number of the polytope vertices already met by the moving plane Q by the time it is a distance h from its initial

position (and similarly for $E(h)$ and $F(h)$). The sum $S(h)$ may change only when Q passes through the vertices $v_1, \ldots, v_n$. When v_1 is in Q, the intersection $Q \cap \partial X$ is a face, an edge or a vertex. From this we conclude that at that moment $S(h) = 1$. Of the same form is the intersection $Q \cap \partial X$ when v_n is in Q. It follows that when Q passes through v_n the sum $S(h)$ increases by 1. When Q passes through a vertex v_i where $2 \leqslant i \leqslant n-1$, all the edges and vertices of X lying at that time in Q form either a simple circuit or one or several simple nonclosed broken lines (possibly, degenerating into vertices). Suppose, for example, that when Q passes through the vertex v_2 there lies in Q a nonclosed broken line C which has α edges and β vertices. Then $\beta = \alpha + 1$. Further, let γ edges and δ faces go upwards from that broken line. Then $\delta = \gamma + 1$. Therefore when Q passes through v_2 the sum $S(h)$ changes by $\beta - (\alpha + \gamma) - \delta = (\alpha + 1) - (\alpha + \gamma) - (\gamma + 1) = 0$, i.e. it retains its value. If C is closed, then $\beta = \alpha$ and $\delta = \gamma$, and $S(h)$ again remains unaltered. This happens when Q passes through all the vertices v_i, $2 \leqslant i \leqslant n-1$. Therefore $\chi(\partial X) = 2$.

18. From the statement of the problem, Euler's formula and from (7.10) and (7.11) we get in turn

$$E = \frac{V(V-1)}{2}, \quad F = 2 - V + E, \quad 3F \leqslant 2E.$$

Hence

$$3\left[2 - V + \frac{V(V-1)}{2}\right] \leqslant V(V-1)$$

or

$$V^2 - 7V + 12 \leqslant 0.$$

Among integers only $V = 3$ and $V = 4$ are the solutions to these inequalities. Since a polytope cannot have three vertices, $V = 4$, i.e. the polytope is a tetrahedron.

19. The dual statement is as follows: if every two faces of a polytope have a common side, then the polytope itself is a tetrahedron.

20. Let the number of sign reversals round each vertex be at least 4. Let N denote the sum of these numbers taken over all the vertices. Under the hypothesis

$$N \geqslant 4V. \tag{S.3}$$

Consider also the number of sign reversals in tracing the boundary of each face and the sum of all such numbers. Since one sign reversal is connected only with one pair of edges that have a vertex in common and are marked with different signs, the total number of reversals in tracing all the faces is equal to the total number of reversals in tracing all the vertices, i.e. to N. The number of sign reversals in tracing an m-gonal face is obviously even and is at most m. Therefore

$$N \leqslant 4F_4 + 4F_5 + 6F_6 + 6F_7 + \ldots \tag{S.4}$$

Using inequalities (S.3) and (S.4) and formulas (7.1) and (7.10) we get

$$\begin{aligned}4V &\leqslant N \leqslant 4F_4+4F_5+6F_6+6F_7\\ &+\ldots \leqslant 2F_3+4F_4+4F_5+6F_6+6F_7\\ &+\ldots = 2(3F_3+4F_4+5F_5+\ldots)-4(F_3+F_4+F_5+\ldots)\\ &=4E-4F=4V-8,\end{aligned}$$

which is impossible.

The dual statement is as follows: there is a polytope face such that in tracing this face the number of sign reversals is at most 2.

21. It follows from $3F \leqslant 2E$ that $E \geqslant 15/2$, i.e., that $E \geqslant 8$. From Euler's formula we get $V = E - 3$. Then the dual inequality $3V \leqslant 2E$ yields $3(E-3) \leqslant 2E$ or $E \leqslant 9$. So only two values are allowed, $E = 8$ and $E = 9$. In the first case Euler's formula yields $V = 5$ and in the second $V = 6$. A polytope with $F = 5$, $E = 8$, and $V = 5$ exists, it is a quadrangular pyramid. A polytope having $F = 5$, $E = 9$, and $V = 6$ is, for example, a triangular prism.

The dual problem is stated as follows: a convex polytope has 5 vertices, how many faces and how many edges may it have?

22. In the formula $2V - 2E + 2F = 4$ we change all the terms of the left-hand side using (7.7), (7.9) and (7.8):

$$2\sum_{i\geqslant 3} V_i - \sum_{i\geqslant 3} iV_i + 2\sum_{i\geqslant 3} F_i = 4. \tag{S.5}$$

We multiply both sides of (S.5) by n and add the equation obtained term by term to

$$2\sum_{i\geqslant 3} iV_i - 2\sum_{i\geqslant 3} iF_i = 0.$$

Then

$$\sum_{i\geqslant 3} (2n-ni+2i)V_i + 2\sum_{i\geqslant 3}(n-i)F_i = 4n.$$

23. Assign to each vertex of each face of the polytope the number 1/3 as its "weight". Then the sum of the weights taken over all faces and over all their vertices is V, i.e. equal to the number of vertices V of the polytope. It follows from the statement of the problem and (7.13) that the polytope has pentagonal faces. Find the sum of the weights taken over all the penta- and hexagonal faces. Suppose that no pentagonal face touches either a pentagonal or a hexagonal face. In view of this assumption the above sum taken over all the pentagonal faces is $\frac{1}{3}\cdot 3\cdot 5F_5 = 5F_5$. But the hexagonal faces may touch one another. The corresponding sum is thus $\frac{1}{3}\cdot 6F_6 = 2F_6$. Therefore we get

$$5F_5 + 2F_6 < V, \tag{S.6}$$

with the inequality being strict since not all the weights of the vertices of n-gonal faces are taken into account for $n \geqslant 7$. On

the other hand, the equality (7.16) for $n = 7$ and $V = V_3$ yields

$$4F_5 + 2F_6 - 2F_8 - 4F_9 - \ldots = 28 + V$$

or

$$4F_5 + 2F_6 \geqslant 28 + V,$$

which contradicts (8.6).

24. 6.

25. 166.

26. The Euler characteristic of the figure is equal to 1.

27. The Euler characteristic of the figure is

$$\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \ldots + (-1)^{n+1}\binom{n}{n} = 1.$$

28. For any partition of a sphere into strictly convex polygons all the relations introduced in Sec. 7 hold. It follows from (7.13) therefore, in particular, that a foot-ball cannot be made from hexagonal pieces alone.

29. There is such a point. Indeed, if a third major circle passed through every intersection point of two major circles, then (taking into consideration the statement of the problem) we would obtain a partition of the sphere into strictly convex faces, each vertex of the partition being of degree $\geqslant 6$, which contradicts inequality (7.14).

30. For example, the boundary of any convex polytope is such a figure.

31. Set $A = \bigcup_{i=1}^{3} A_i$, $B = \bigcap_{i=1}^{3} A_i$. Then $\chi(A) = 3 - 3 + \chi(B) = \chi(B)$. Under the hypothesis we have $\chi(A) = 1$, therefore $\chi(B) = 1$, and this immediately yields the required statement.

32. Set $A = \bigcup_{i=1}^{4} A_i$, $B = \bigcap_{i=1}^{4} A_i$. Then

$$\chi(A) = \binom{4}{1} - \binom{4}{2} + \binom{4}{3} - \chi(B) = 2 - \chi(B).$$

It is required to prove that $B = \varnothing$. To do this it suffices to verify that $\chi(B) \neq 0$. Assume the contrary. Let $\chi(B) = 0$. Then $\chi(A) = 2$. Since every two polygons have a point in common, the figure A is connected. Hence $c(A) = 1$, and from (12.1) we get $c^*(A) = 0$. This last equation means that A coincides with the plane. But this is impossible since the figure A is bounded and the plane is not.

33. Given polygons A_1, A_2, A_3, and A_4, prove that the intersection of every three of them is nonempty, and that the required statement follows from the preceding problem. Set $A = \bigcup_{i=1}^{3} A_i$, $B = \bigcap_{i=1}^{3} A_i$. Then, just as in Problem 31, $\chi(A) = \chi(B)$. Under the hypothesis $c(A) = 1$ and $c^*(A) = 1$. From (12.1) we therefore have $\chi(A) = 1$, and therefore $\chi(B) = 1$. Hence $B \neq \varnothing$. Similarly it can be verified that the intersection of all the triples of polygons is nonempty.

35. Into $\frac{1}{2}(n^2 - 3n + 4)$ parts.

36. Set $A = \bigcup_{i=1}^{n} A_i$, where A_i are segments. Since $c(A) \leqslant n$ and $c^*(A) \geqslant 1$, from (12.1) we get $\chi(A) \leqslant n$. The inequality on the left is proved by induction on the number of segments, n. For $n = 1$ it is obvious. Suppose it is already proved for all $n \leqslant m$. Prove it for $n = m+1$. Let $B = \bigcup_{i=1}^{m+1} A_i$. Under the hypothesis we have $\chi\left(\bigcup_{i=1}^{m} A_i\right) \geqslant \frac{m(3-m)}{2}$, therefore

$$\chi(B) = \chi\left(\bigcup_{i=1}^{m} A_i\right) + \chi(A_{m+1}) - \chi\left[A_{m+1} \cap \left(\bigcup_{i=1}^{m} A_i\right)\right]$$

$$\geqslant \frac{m(3-m)}{2} + 1 - \chi\left[\bigcup_{i=1}^{m} (A_{m+1} \cup A_i)\right].$$

Considering that $\chi\left[\bigcup_{i=1}^{m} (A_{m+1} \cap A_i)\right] \leqslant m$ we get

$$\chi(B) \geqslant \frac{m(3-m)}{2} + 1 - m = \frac{(m+1)[3-(m+1)]}{2}.$$

37. Given strictly convex polygons $A_1, A_2, \ldots, A_5$, suppose that every four of them have an empty intersection. By Theorem 3 every four polygons and hence all the five polygons cover a sphere S. Then

$$2 = \chi(S) = \binom{5}{1} - \binom{5}{2} + \binom{5}{3} = 5.$$

The contradiction obtained proves the required statement.

38. Let every four polygons cover a sphere S. By Theorem 3 then every three of them have a nonempty intersection and every four have an empty intersection. Let there be m polygons ($m \geqslant 5$) all together. Then

$$2 = \chi(S) = \binom{m}{1} - \binom{m}{2} + \binom{m}{3} = 1 + \binom{m-1}{3}.$$

Hence $\binom{m-1}{3} = 1$ or $m = 4$, which is impossible.

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This booklet gives proofs of Euler's famous formula for convex polytopes and of its analogues for other figures (planes, spaces and polygons). The formulas bring the reader naturally to the notion of Euler's characteristic. Two definitions of the notion are given and their equivalence is proved. The part played by the Euler characteristic in different geometrical problems, i.e. in the decomposition of planes and spaces, in calculating areas, in coverings of spheres, is discussed.

The book is intended for senior pupils, junior college and university students and all lovers of mathematics.

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